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Analysis of an inviscid zero-Mach number system in endpoint Besov spaces for finite-energy initial data

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Abstract

The present paper is the continuation of work [18], devoted to the study of an inviscid zero-Mach number system in the framework of *endpoint* Besov spaces of type $B^s_{\infty,r}(\mathbb{R}^d)$, $r \in [1, \infty]$, $d \geq 2$, which can be embedded in the Lipschitz class $C^{0,1}$. In particular, the largest case $B^1_{\infty,1}$ and the case of Hölder spaces $C^{1,\alpha}$ are taken into account.

The local in time well-posedness of this system is proved, under an additional finite-energy hypothesis on the initial data. The key to get this result is new a priori estimates for parabolic equations with variable coefficients in endpoint spaces $B^s_{\infty,r}(\mathbb{R}^d)$, which are of independent interest.

In the special case of space dimension $d = 2$, we are able to give a lower bound for the lifespan, such that the solutions tend to be globally defined when the initial inhomogeneity is small. There, we will show refined a priori estimates in endpoint Besov spaces for transport equations.

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1. Introduction

In the present paper we will study the following *inviscid zero-Mach number system*:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) & = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla \Pi & = 0, \\ \operatorname{div}(v + \kappa \rho^{-1} \nabla \rho) & = 0, \end{cases} \tag{1}$$

where $\rho = \rho(t,x) \in \mathbb{R}^+$ stands for the mass density, $v = v(t,x) \in \mathbb{R}^d$ for the velocity field and $\Pi = \Pi(t,x) \in \mathbb{R}$ for the unknown pressure. The positive heat-conducting coefficient $\kappa = \kappa(\rho)$ depends smoothly on its variable. The time variable t and the space variable x belong to $\mathbb{R}^+$ (or to $[0,T]$) and $\mathbb{R}^d$, $d \geq 2$, respectively.

This model derives from the full compressible, heat-conducting and inviscid system as the Mach number tends to vanish (see e.g. [1,19,24,25,30]). In particular, this singular low-Mach number limit process is rigorously justified in Alazard [1] for smooth enough solutions. We refer to the introduction of [18] or the previous literature for more details on the derivation of the system.

Interestingly, System (1) can also describe, for instance, the motion of a two-component incompressible inviscid mixture with diffusion effects between these two components. We refer to e.g. [20] for more physical backgrounds.

Notice that if we take simply $\kappa \equiv 0$ (i.e. we have no heat conduction), then System (1) reduces to the density-dependent Euler equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla \Pi = 0, \\ \operatorname{div} v = 0. \end{cases} \tag{2}$$

We refer to [2,12,13], among other works, for some well-posedness results for System (2). Let us just mention here that, in [14], Danchin adopted mainly the functional framework of Besov spaces $B^s_{p,r}$, $1 < p < +\infty$, which can be embedded in the set of globally Lipschitz functions. There he considered the case of finite-energy initial data, or the case when $p \in [2,4]$, or the case of small inhomogeneity. All the assumptions are, roughly speaking, due to the control of (the low frequencies of) the pressure term. In [15], Danchin and Fanelli treated the endpoint case $B^s_{\infty,r}$, giving, besides, a lower bound for the lifespan of solutions in the case of space dimension $d = 2$; infinite energy data were considered as well, for which one has to resort to the analysis of the vorticity of the fluid.

When the fluid is supposed to be viscous, instead, System (1) becomes the viscous zero-Mach number system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) & = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) - \operatorname{div}\sigma + \nabla \Pi & = 0, \\ \operatorname{div}(v + \kappa \rho^{-1} \nabla \rho) & = 0, \end{cases} \tag{3}$$

where we defined the viscous stress tensor

$$\sigma = 2\zeta S v + \eta \operatorname{div} v \operatorname{Id}, \quad S v := \frac{1}{2}(\nabla v + (\nabla v)^T), \quad \operatorname{Id}: \text{the identity } \mathbb{R}^d \times \mathbb{R}^d \text{ matrix},$$

with two positive viscous coefficients ζ, η.

The viscous system (3) is the low-Mach number limit system of the full Navier–Stokes equations, and hence it describes for instance the motion of highly subsonic viscous fluid. See [1, 9,11,17,22,23,26,27] and the references therein for further results. Let us just mention that, in [16], Danchin and Liao addressed the well-posedness issue in the general critical Besov spaces $B^{d/p_1}_{p_1,1}(\mathbb{R}^d) \times (B^{d/p_2-1}_{p_2,1}(\mathbb{R}^d))^d$, with technical restrictions on the Lebesgue exponents p_1, p_2. Under a special relationship between the viscous coefficient and the heat-conduction coefficient, which is a sort of remainder of the BD-entropy structure of Bresch and Desjardins for such kind of systems (see e.g. [6–8] and the references therein), Liao showed in [23] the global in time existence of weak solutions and in particular global in time well-posedness result in dimension two.

To our knowledge, there are just few well-posedness results for the inviscid zero-Mach number system (1). For instance, in [4] Beirão da Veiga, Serapioni and Valli proved existence of classical solutions on smooth bounded domains.

In our previous work [18], instead, we investigate the well-posedness in the functional framework of general Besov spaces $B^s_{p,r}(\mathbb{R}^d)$, $p \in [2,4]$, which can be embedded in the Lipschitz function class. There, we reformulated System (1) by introducing a new *divergence-free* velocity field. Similarly, let us immediately perform this invertible change of unknowns here, to introduce the set of equations (see (6) below) we will mainly work on: for the details we refer to [16,18].

For notational simplicity, we introduce three "coefficients", $a = a(\rho)$, $b = b(\rho)$ and $\lambda = \lambda(\rho)$, such that

$$\nabla a = \kappa \nabla \rho = -\rho \nabla b, \quad \lambda = \rho^{-1} > 0, \quad a(1) = b(1) = 0. \tag{4}$$

Then, we introduce the new divergence-free "velocity" u and the new "pressure" π as

$$u := v + \kappa \rho^{-1} \nabla \rho = v - \nabla b, \qquad \pi = \Pi - \kappa \partial_t \rho = \Pi - \partial_t a\,. \tag{5}$$

Therefore, System (1) can be rewritten as the following system for the unknowns (ρ, u, π):

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho - \mathrm{div}\,(\kappa \nabla \rho) = 0, \\ \partial_t u + (u + \nabla b) \cdot \nabla u + \lambda \nabla \pi = h, \\ \mathrm{div}\, u = 0, \end{cases} \tag{6}$$

where the new nonlinear "source" term h reads as

$$h(\rho, u) = \rho^{-1} \mathrm{div}\,(v \otimes \nabla a) = -u \cdot \nabla^2 b - (u \cdot \nabla \lambda) \nabla a - (\nabla b \cdot \nabla \lambda) \nabla a - \mathrm{div}\,(\nabla b \otimes \nabla b)\,. \tag{7}$$

In the above mentioned work [18], we studied the well-posedness of the zero-Mach number system, in its reformulated version (6), in the setting of Besov spaces $B^s_{p,r}(\mathbb{R}^d)$ embedded in the class $C^{0,1}$ of globally Lipschitz functions, that is to say for

$$s > 1 + \frac{d}{p}, \qquad \text{or} \qquad s = 1 + \frac{d}{p} \quad \text{and} \quad r = 1\,. \tag{C}$$

Such a restriction is in fact necessary, essentially due to the transport equation for the velocity field: preserving the initial regularity demands u to be at least locally Lipschitz with respect to the space variable. On the other hand, the non-linear source term h requires the control of this

Besov norm on $\nabla^2\rho$: this is guaranteed by the smoothing effect of the *parabolic equation* for the density. Due to technical reasons, in [18] we had to impose the additional condition

$$p \in [2,4]\,. \tag{8}$$

Indeed, this hypothesis (8) ensures that the "source" term h, composed of quadratic terms, belongs to $L^2(\mathbb{R}^d)$. Inturn, due to the elliptic equation in divergence form for the pressure π

$$\mathrm{div}\,(\lambda\nabla\pi) = \mathrm{div}\,(h - (u + \nabla b)\cdot\nabla u),$$

this hypothesis (8) ensures that also the pressure term $\nabla\pi$ belongs to $L^2(\mathbb{R}^d)$, which gives a control on the low frequencies of the pressure term.

Finally, under conditions (C) and (8), we [18] proved local in time well-posedness of System (6) in general Besov spaces $B^s_{p,r}(\mathbb{R}^d)$, as well as a continuation criterion for its solutions and a bound from below for the lifespan in any space dimension $d \geq 2$.

In the present paper we propose a different study, rather in endpoint Besov spaces $B^s_{\infty,r}$ which still verifies condition (C) (with $p = +\infty$ of course), in the same spirit of work [15]. This functional framework includes, in particular, the case of Hölder spaces of type $C^{1,\alpha}$ and the case of $B^1_{\infty,1}$, which is the largest Besov space embedded in the set of globally Lipschitz functions, and so the largest one in which one can expect to recover well-posedness for our system.

We also add a finite-energy hypothesis on the initial data, which is fundamental in order to control the pressure term, just as the above condition (8) assumed in [18].

Then we are able to prove the local in time well-posedness of System (6) in the adopted endpoint functional framework. The key point of the analysis is the proof of new a priori estimates for parabolic equations with variable coefficients, not necessarily close to a constant, in Besov spaces $B^s_{\infty,r}$, see Proposition 4.1. They state that the parabolic gain of regularity (of two orders as for the heat equations) holds true also in this setting. Roughly speaking, they are obtained by use of a "microlocal" analysis argument: first of all, we localize the parabolic equation in space, in order to work on small sets, where we only have to deal with a non-homogeneous heat equation; then we stick these localized parts together, keeping in the same time the Hölder regularity. We refer to the beginning of Subsection 4.1 for some additional insights about the proof of this result.

The global in time existence of strong solutions to the inviscid zero-Mach number system is still an open problem, even in the simpler case of space dimension $d = 2$. However, similarly as in [15], we are able to move a first step in this direction: by establishing an explicit lower bound for the lifespan of the solutions in dimension $d = 2$, we show that planar flows tend to be globally defined if the initial density is "close" (in an appropriate sense) to a constant state. Such a lower bound improves the one stated in [18] which holds for any space dimension. It can be obtained resorting to arguments similar as in Vishik [28] and Hmidi and Keraani [21]: the scalar vorticity satisfies a transport equation, and then one aims at bounding it *linearly* with respect to the velocity field. As we encounter here non-solenoidal velocity field v, one has to bound the vorticity linearly in v and $\mathrm{div}\,v$ (see Proposition 4.4). Since the potential part of v just depends on the density term ρ (see $(1)_3$), the parabolic effect gives enough regularity to control $\mathrm{div}\,v$.

Let us conclude the introduction by pointing out that we decided to adopt the present functional framework, i.e. $B^s_{\infty,r} \cap L^2$, just for simplicity and clarity of exposition. Actually, combining the techniques of [18] with the ones in [14], it's easy to see that our results can be extended to any space $B^s_{p,r}$ which satisfies condition (C) for any $1 < p \leq +\infty$.

Before going on, we give an overview of the paper.

Next section is devoted to the statement of our two main results.

In Section 3 we briefly present the tools we are going to use in the analysis, namely Littlewood–Paley decomposition and paradifferential calculus, while in Section 4 we prove fundamental a priori estimates for parabolic equations (Proposition 4.1) and for transport equations (Proposition 4.4) in endpoint Besov spaces.

Finally, Section 5 contains the proofs of our two results.

2. Main results

As explained in the introduction, in the sequel we will deal with system (6)–(7) in endpoint Besov spaces $B^s_{\infty,r}(\mathbb{R}^d)$, $d \geq 2$ with the indices $s \in \mathbb{R}$ and $r \in [1, +\infty]$ satisfying (C) (for $p = +\infty$), i.e.

$$s > 1 \qquad \text{or} \qquad s = r = 1\,. \tag{9}$$

Recall that this is sufficient to guarantee the embedding $B^s_{\infty,r}(\mathbb{R}^d) \hookrightarrow C^{0,1}(\mathbb{R}^d)$.

In order to ensure the velocity field u to belong to $B^s_{\infty,r}(\mathbb{R}^d)$, the source term h in the velocity equation, which involves two derivatives of the density $\nabla^2\rho$, should be in the same space. This will be provided by new a priori estimates for parabolic equations in endpoint Besov spaces $B^s_{\infty,r}$ (see Proposition 4.1 below), which allow the gain of two orders of regularity for the density as time goes by. For this reason we take the initial inhomogeneity $\varrho_0 := \rho_0 - 1 \in B^s_{\infty,r}(\mathbb{R}^d)$, and hence we will get the density in the so-called Chemin–Lerner spaces $\widetilde{L}^\infty([0,T]; B^s_{\infty,r}(\mathbb{R}^d)) \cap \widetilde{L}^1([0,T]; B^{s+2}_{\infty,r}(\mathbb{R}^d))$. We refer to Definition 3.6 for the precise definition of these time-dependent Besov spaces.

Moreover, in order to avoid vacuum regions, we will always suppose the initial density to satisfy

$$0 < \rho_* \leq \rho_0(x) \leq \rho^*\,, \quad x \in \mathbb{R}^d.$$

By applying maximum principle on the parabolic equation $(6)_1$, one gets a priori that the density ρ (if it exists on the time interval $[0,T]$) keeps the same upper and lower bounds as the initial density ρ_0:

$$0 < \rho_* \leq \rho(t,x) \leq \rho^*, \quad \forall t \in [0,T],\ x \in \mathbb{R}^d.$$

Hence, applying the divergence operator to Equation $(6)_2$ gives an elliptic equation for π of the form

$$\operatorname{div}(\lambda\nabla\pi) = \operatorname{div}(h - v\cdot\nabla u)\,, \qquad \text{with} \quad \lambda = \lambda(\rho) \geq \lambda_* := (\rho^*)^{-1} > 0\,. \tag{10}$$

By a result in [14], we hence have a priori energy estimates for $\nabla\pi$ (independently on ρ):

$$\lambda_*\|\nabla\pi\|_{L^2} \leq \|h - v\cdot\nabla u\|_{L^2}\,.$$

This gives low frequency informations for $\nabla\pi$.

One then considers the following energy estimates. First of all, the mass conservation law (6)$_1$ entails (provided $u \in L^\infty_T(L^\infty)$)

$$\frac{1}{2}\int_{\mathbb{R}^d} |\rho(t)-1|^2 \mathrm{d}x + \int_0^t\int_{\mathbb{R}^d} \kappa|\nabla\rho|^2 \mathrm{d}x\,\mathrm{d}t' = \frac{1}{2}\|\rho_0-1\|^2_{L^2(\mathbb{R}^d)}. \tag{11}$$

Next we rewrite the momentum conservation law (6)$_2$ into

$$\rho\partial_t u + \rho v\cdot\nabla u + \nabla\pi = \mathrm{div}\,(v\otimes\nabla a) \equiv u\cdot\nabla^2 a + \Delta b\nabla a + \nabla b\cdot\nabla^2 a. \tag{12}$$

Then, using equation (1)$_1$ and $\mathrm{div}\, u = 0$, taking the $L^2(\mathbb{R}^d)$ scalar product between (12) and u entails

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} \rho|u|^2\,\mathrm{d}x \equiv \int_{\mathbb{R}^d} (\rho\partial_t u + \rho v\cdot\nabla u + \nabla\pi)\cdot u\,\mathrm{d}x = \Big\langle u\cdot\nabla^2 a + \Delta b\nabla a + \nabla b\cdot\nabla^2 a, u\Big\rangle_{L^2(\mathbb{R}^d)}. \tag{13}$$

Recalling the definitions of a and b in (4), one bounds the right-hand side of (13) by (up to a multiplicative constant depending on ρ_* and ρ^*)

$$\left(\Theta'(t)\,\|u\|^2_{L^2} + \|\nabla\rho\|^2_{L^2}\right), \quad \text{with}$$

$$\Theta(t) := \int_0^t \left(\|\nabla\rho\|^2_{L^\infty} + \|\nabla\rho\|^4_{L^\infty} + \|\nabla^2\rho\|_{L^\infty} + \|\nabla^2\rho\|^2_{L^\infty}\right)\mathrm{d}t'\,.$$

Hence if $(\rho_0-1, u_0) \in L^2(\mathbb{R}^d)$ and $\Theta(T) < +\infty$ (this will be ensured by Besov regularity), then we gather

$$u, \rho-1 \in L^\infty_T(L^2(\mathbb{R}^d)), \quad \nabla\rho \in L^2_T(L^2(\mathbb{R}^d))$$

and hence $h \in L^1([0,T]; L^2(\mathbb{R}^d))$, $\nabla\pi \in L^1([0,T]; L^2(\mathbb{R}^d))$.

To conclude, we have the following local-in-time wellposedness result for System (6).

Theorem 2.1. *Let $d \geq 2$ be an integer and take $s \in \mathbb{R}$ and $r \in [1,+\infty]$ satisfying condition (9).*

Suppose that the initial data (ρ_0, u_0) fulfill

$$\rho_0 - 1\,,\, u_0 \in B^s_{\infty,r}(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)\,, \qquad \rho_0 \in [\rho_*, \rho^*]\,, \qquad \mathrm{div}\, u_0 = 0\,. \tag{14}$$

Then there exist a positive time T and a unique solution $(\rho, u, \nabla\pi)$ to System (6) such that $(\varrho, u, \nabla\pi) := (\rho-1, u, \nabla\pi)$ belongs to the space $E^s_r(T)$, defined as the set of triplet $(\varrho, u, \nabla\pi)$ such that

$$\begin{cases} \varrho \in \widetilde{C}\big([0,T]; B^s_{\infty,r}(\mathbb{R}^d)\big) \cap \widetilde{L}^1\big([0,T]; B^{s+2}_{\infty,r}(\mathbb{R}^d)\big) \cap C\big([0,T]; L^2(\mathbb{R}^d)\big), \\ \nabla\varrho \in L^2\big([0,T]; L^2(\mathbb{R}^d)\big), \quad \rho_* \le \varrho + 1 \le \rho^* \ \text{on } [0,T]\times\mathbb{R}^d, \\ u \in \widetilde{C}\big([0,T]; B^s_{\infty,r}(\mathbb{R}^d)\big)^d \cap C\big([0,T]; L^2(\mathbb{R}^d)\big)^d, \\ \nabla\pi \in \widetilde{L}^1\big([0,T]; B^s_{\infty,r}(\mathbb{R}^d)\big)^d \cap L^1\big([0,T]; L^2(\mathbb{R}^d)\big)^d, \end{cases} \tag{15}$$

with $\widetilde{C}_w([0,T]; B^s_{p,r}(\mathbb{R}^d))$ *if* $r = +\infty$.

Remark 2.2. Let us remark the well-posedness result for the original system (1). According to the change of variables (5), one knows

$$u = \mathcal{P}v, \quad \nabla b = \mathcal{Q}v, \qquad \text{where} \quad \widehat{\mathcal{Q}v}(\xi) = -(\xi/|\xi|^2)\,\xi\cdot\widehat{v}(\xi), \quad \mathcal{P}v = v - \mathcal{Q}v.$$

Hence, for the original system (1), if the initial datum (ρ_0, v_0) satisfies

$$0 < \rho_* \le \rho_0 \le \rho^*, \quad \nabla b(\rho_0) = \mathcal{Q}v_0, \qquad \rho_0 - 1, \mathcal{P}v_0 \in B^s_{\infty,r}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d),$$

with s, r satisfying (9), then there exist a $T > 0$ and a unique solution $(\rho, v, \nabla\Pi)$ to System (1) such that $\rho_* \le \rho \le \rho^*$ and

$$\begin{cases} \varrho = \rho - 1 \in \widetilde{C}\big([0,T]; B^s_{\infty,r}(\mathbb{R}^d)\big) \cap \widetilde{L}^1\big([0,T]; B^{s+2}_{\infty,r}(\mathbb{R}^d)\big) \cap C\big([0,T]; L^2(\mathbb{R}^d)\big), \\ \nabla\varrho \in L^2\big([0,T]; L^2(\mathbb{R}^d)\big), \\ \mathcal{P}v \in \widetilde{C}\big([0,T]; B^s_{\infty,r}(\mathbb{R}^d)\big) \cap C\big([0,T]; L^2(\mathbb{R}^d)\big), \quad v \in \widetilde{C}\big([0,T]; B^{s-1}_{\infty,r}(\mathbb{R}^d)\big), \\ \nabla\Pi \in \widetilde{L}^1\big([0,T]; B^s_{\infty,r}(\mathbb{R}^d)\big), \end{cases}$$

with $\widetilde{C}_w([0,T]; B^s_{p,r}(\mathbb{R}^d))$ if $r = +\infty$.

Remark 2.3. As said in the introduction, we can replace the Besov space $B^s_{\infty,r}(\mathbb{R}^d)$ in Theorem 2.1 by any general Besov space $B^s_{p,r}(\mathbb{R}^d)$, $p \in]1,+\infty]$ such that condition (C) is fulfilled.

The proof is quite standard, and it goes along the lines of the one in [18], with suitable modifications corresponding to the finite energy conditions. One can refer also to paper [14], where an analogous result is proved for the density-dependent Euler equations.

If $\rho \equiv 1$, System (6) becomes the classical Euler system. For this system, the global-in-time existence issue in dimension $d = 2$ has been well-known since 1933, due to the pioneering work [29] of Wolibner. For non-homogeneous perfect fluids, see system (2), it's still open if its 2-D solutions exist globally in time. However, in [15] it was proved that, for initial densities close to a constant state, the lifespan of the corresponding solutions tends to infinity. We have an analogue result for our system and for simplicity let us just state it in the following theorem for the case with the initial inhomogeneity smaller than "1".

Theorem 2.4. *Let* $d = 2$, *and let us assume the hypotheses of Theorem 2.1 and* $\|\rho_0 - 1\|_{B^1_{\infty,1}(\mathbb{R}^2)} \le 1$.

Then there exist a constant $c > 0$ *(depending only on* ρ_*, ρ^*, s, r*) such that the lifespan of the solution to System (6), given by Theorem 2.1, is bounded from below by the quantity*

$$\frac{c}{\Gamma_0}\log\left(\frac{c}{\Gamma_0^2}\log\left(1+\frac{c}{\|\rho_0-1\|_{B^1_{\infty,1}(\mathbb{R}^2)}}\right)\right), \tag{16}$$

where we defined $\Gamma_0 = 1+\|\rho_0-1\|^2_{L^2(\mathbb{R}^2)}+\|u_0\|_{L^2\cap B^1_{\infty,1}(\mathbb{R}^2)}$.

Remark 2.5. We can just consider in (16) the limit Besov space norm $B^1_{\infty,1}(\mathbb{R}^2)$, instead of the general Besov norm $B^s_{\infty,r}(\mathbb{R}^2)$. In fact, similar as in the proof of the continuation criterion in [18], by classical commutator estimates and product estimates (see Proposition 3.4 and Proposition 3.5 below), one knows that if, on the time interval $[0,T^*]$, $T^*<+\infty$, one has

$$\|(\nabla\rho,u)\|_{L^\infty_{T^*}(L^\infty(\mathbb{R}^2))}+\int_0^{T^*}\left(1+\|\nabla u\|^2_{L^\infty(\mathbb{R}^2)}+\|\nabla^2\varrho\|^2_{L^\infty(\mathbb{R}^2)}+\|\nabla\pi\|_{L^\infty(\mathbb{R}^2)}\right)\mathrm{d}t<+\infty,$$

then the solution (ϱ,u) with the initial data $(\varrho_0,u_0)\in B^s_{\infty,r}(\mathbb{R}^2)$ will be well defined in the solution space $E^s_r(T^*)$. On the other side, the above continuation condition can be ensured if one only has the solution defined in the limit space $E^1_1(T^*)$.

Before going on, let us introduce some notations. We agree that in the sequel, C always denotes some "harmless" constant depending only on d, s, r, ρ_*, ρ^*, unless otherwise defined. Notation $A\lesssim B$ means $A\le CB$ and $A\sim B$ says A equals to B, up to a constant factor. For notational convenience, we denote

$$\varrho=\rho-1.$$

3. A short review of Fourier analysis

In this section, we recall some definitions and results in Fourier analysis which will be used in this paper. Unless otherwise specified, all the presentations in this section have been proved in [3], Chapter 2.

Firstly, let's recall the Littlewood–Paley decomposition on the whole space $\mathbb{R}^d$. Fix a smooth radial function χ supported in the ball $B(0,\frac{4}{3})$, such that it equals to 1 in a neighborhood of $B(0,\frac{3}{4})$ and is nonincreasing over $\mathbb{R}_+$. Define $\varphi(\xi)=\chi(\frac{\xi}{2})-\chi(\xi)$. The non-homogeneous dyadic blocks $(\Delta_j)_{j\in\mathbb{Z}}$ are defined by[2]

$$\Delta_j u := 0 \ \text{ if } \ j\le -2,\quad \Delta_{-1}u:=\chi(D)u \quad\text{and}\quad \Delta_j u:=\varphi(2^{-j}D)u \ \text{ if } \ j\ge 0.$$

We also introduce the following low frequency cut-off operators:

$$S_j u := \chi(2^{-j}D)u=\sum_{j'\le j-1}\Delta_{j'}u \quad\text{for}\quad j\ge 0,\quad\text{and}\quad S_j u\equiv 0\quad\text{for}\quad j<0.$$

One hence defines non-homogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ as follows:

2 In what follows we agree that $f(D)$ stands for the pseudo-differential operator $u\mapsto\mathcal{F}^{-1}(f(\xi)\mathcal{F}u(\xi))$.

Definition 3.1. Let $u \in \mathcal{S}'$, $s \in \mathbb{R}$, $(p,r) \in [1,\infty]^2$. We set

$$\|u\|_{B^s_{p,r}(\mathbb{R}^d)} := \left(\sum_{j \geq -1} 2^{rjs} \|\Delta_j u\|^r_{L^p(\mathbb{R}^d)} \right)^{\frac{1}{r}} \quad \text{if } r < \infty \quad \text{and}$$

$$\|u\|_{B^s_{p,\infty}} := \sup_{j \geq -1} \left(2^{js} \|\Delta_j u\|_{L^p(\mathbb{R}^d)} \right).$$

The space $B^s_{p,r}(\mathbb{R}^d)$ is the subset of tempered distributions u such that $\|u\|_{B^s_{p,r}(\mathbb{R}^d)}$ is finite.

Recall that, for all $s \in \mathbb{R}$, we have the equivalence $H^s \equiv B^s_{2,2}$, while for all $s \in \mathbb{R}_+ \backslash \mathbb{N}$, the space $B^s_{\infty,\infty}$ is actually the Hölder space C^s. If $s \in \mathbb{N}$, instead, we set $C^s_* := B^s_{\infty,\infty}$, to distinguish it from the space C^s of the differentiable functions with continuous partial derivatives up to the order s. Moreover, the strict inclusion $C^s_b \hookrightarrow C^s_*$ holds, where C^s_b denotes the subset of C^s functions bounded with all their derivatives up to the order s. Finally, for $s < 0$, the "negative Hölder space" C^s is defined as the Besov space $B^s_{\infty,\infty}$.

For spectrally localized functions, one has the following Bernstein's inequalities:

Lemma 3.2. *There exists a $C > 0$ such that, for any $k \in \mathbb{Z}^+$, $\lambda \in \mathbb{R}^+$, $(p,q) \in [1,\infty]^2$ with $p \leq q$, then*

$$\operatorname{Supp} \widehat{u} \subset B(0,\lambda) \Longrightarrow \|u\|_{L^q(\mathbb{R}^d)} \leq C \lambda^{d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p(\mathbb{R}^d)};$$

$$\operatorname{Supp} \widehat{u} \subset \{\xi \in \mathbb{R}^d \,/\, \lambda \leq |\xi| \leq 2\lambda\} \Longrightarrow C^{-k-1}\lambda^k \|u\|_{L^p(\mathbb{R}^d)} \leq \|\nabla^k u\|_{L^p(\mathbb{R}^d)} \leq C^{k+1}\lambda^k \|u\|_{L^p(\mathbb{R}^d)}.$$

We remark explicitly that by previous lemma one has, for any $f \in L^2(\mathbb{R}^d)$,

$$\|\Delta_{-1} f\|_{L^\infty(\mathbb{R}^d)} \leq C \|\Delta_{-1} f\|_{L^2(\mathbb{R}^d)} \leq C \|f\|_{L^2(\mathbb{R}^d)}.$$

One also has the following embedding and interpolation results:

Proposition 3.3. *$B^{s_1}_{p_1,r_1}(\mathbb{R}^d)$ is continuously embedded in $B^{s_2}_{p_2,r_2}(\mathbb{R}^d)$ whenever $1 \leq p_1 \leq p_2 \leq \infty$ and*

$$s_2 < s_1 - d/p_1 + d/p_2 \quad \text{or} \quad s_2 = s_1 - d/p_1 + d/p_2 \text{ and } 1 \leq r_1 \leq r_2 \leq \infty.$$

Moreover, one has the following interpolation inequality:

$$\|\varrho\|_{B^{s'}_{\infty,r}(\mathbb{R}^d)} \leq C \|\varrho\|^{\frac{s+2-s'}{2}}_{B^s_{\infty,r}(\mathbb{R}^d)} \|\varrho\|^{\frac{s'-s}{2}}_{B^{s+2}_{\infty,r}(\mathbb{R}^d)}, \quad \forall s' \in [s, s+2] \tag{17}$$

$$\|\nabla \pi\|_{B^{s''}_{\infty,r}(\mathbb{R}^d)} \leq C \|\nabla \pi\|^{\alpha}_{L^2(\mathbb{R}^d)} \|\nabla \pi\|^{1-\alpha}_{B^s_{\infty,r}(\mathbb{R}^d)} \quad \forall s'' < s, \text{ with } \alpha = \alpha(s'') \in (0,1). \tag{18}$$

One also has the following classical commutator estimate:

Proposition 3.4. *If $s > 0$, $r \in [1, \infty]$, then there exists a constant C depending only on d, s, r such that*

$$\int_0^t \left\| 2^{js} \left\| [\varphi, \Delta_j] \cdot \nabla\psi \right\|_{L^\infty} \right\|_{\ell^r} dt' \le C \int_0^t \left(\|\nabla\varphi\|_{L^\infty} \|\psi\|_{B^s_{\infty,r}} + \|\nabla\varphi\|_{B^{s-1}_{\infty,r}} \|\nabla\psi\|_{L^\infty} \right) dt'. \tag{19}$$

Moreover if $s \in (0, 1)$ (or $s \in (-1, 1)$ if $\operatorname{div}\varphi = 0$), then there holds

$$\int_0^t \left\| 2^{js} \left\| [\varphi, \Delta_j] \cdot \nabla\psi \right\|_{L^\infty} \right\|_{\ell^r} dt' \le C \int_0^t \|\nabla\varphi\|_{L^\infty} \|\psi\|_{B^s_{\infty,r}} dt'.$$

Let us recall Bony's paraproduct decomposition (first introduced in [5]):

$$uv = T_u v + T_v u + R(u, v), \tag{20}$$

where we defined the paraproduct operator T and the remainder R as

$$T_u v := \sum_j S_{j-1} u \Delta_j v \quad \text{and} \quad R(u, v) := \sum_j \sum_{|j'-j|\le 1} \Delta_j u \, \Delta_{j'} v \, .$$

These operators enjoy the following continuity properties in the class of Besov spaces.

Proposition 3.5. *For any $(s, p, r) \in \mathbb{R} \times [1, \infty]^2$ and $t > 0$, the paraproduct operator T maps $L^\infty \times B^s_{p,r}$ in $B^s_{p,r}$, and $B^{-t}_{\infty,\infty} \times B^s_{p,r}$ in $B^{s-t}_{p,r}$. Moreover, the following estimates hold:*

$$\|T_u v\|_{B^s_{p,r}} \le C \|u\|_{L^\infty} \|\nabla v\|_{B^{s-1}_{p,r}} \quad \text{and} \quad \|T_u v\|_{B^{s-t}_{p,r}} \le C \|u\|_{B^{-t}_{\infty,\infty}} \|\nabla v\|_{B^{s-1}_{p,r}}.$$

For any (s_1, p_1, r_1) and (s_2, p_2, r_2) in $\mathbb{R} \times [1, \infty]^2$ such that $s_1 + s_2 > 0$, $1/p := 1/p_1 + 1/p_2 \le 1$, the remainder operator R maps $B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$ in $B^{s_1+s_2}_{p,r}$ with $1/r := \min\{1/r_1 + 1/r_2, 1\}$.

If $s > 0$, then one can easily bound the product uv as follows:

$$\|uv\|_{B^s_{p,r}} \le C \big(\|u\|_{L^\infty} \|v\|_{B^s_{p,r}} + \|u\|_{B^s_{p,r}} \|v\|_{L^\infty} \big). \tag{21}$$

When solving evolutionary PDEs in Besov spaces, we have to localize the equations by Littlewood–Paley decomposition. So we will have estimates for the Lebesgue norm of each dyadic block before performing integration in time. This leads to the following definition, introduced for the first time in paper [10] by Chemin and Lerner.

Definition 3.6. For $s \in \mathbb{R}$, $(q, p, r) \in [1, +\infty]^3$ and $T \in [0, +\infty]$, we set

$$\|u\|_{\widetilde{L}^q_T(B^s_{p,r})} := \left\| \left(2^{js} \|\Delta_j u(t)\|_{L^q_T(L^p)} \right)_{j \ge -1} \right\|_{\ell^r}.$$

We also set $\widetilde{C}_T(B^s_{p,r}) = \widetilde{L}^\infty_T(B^s_{p,r}) \cap C([0, T]; B^s_{p,r})$.

The relation between these classes and the classical $L^q_T(B^s_{p,r})$ can be easily recovered by Minkowski's inequality:

$$\begin{cases} \|u\|_{\widetilde{L}^q_T(B^s_{p,r})} \leq \|u\|_{L^q_T(B^s_{p,r})} & \text{if } \ q \leq r \\ \|u\|_{\widetilde{L}^q_T(B^s_{p,r})} \geq \|u\|_{L^q_T(B^s_{p,r})} & \text{if } \ q \geq r\,. \end{cases}$$

Combining the above Proposition 3.5 with Bony's decomposition (20), we easily get the following product estimates in Chemin–Lerner space:

Corollary 3.7. *There exists a constant C depending only on d, s, p, r such that*

$$\|uv\|_{\widetilde{L}^q_T(B^s_{p,r})} \leq C\left(\|u\|_{L^{q_1}_T(L^\infty)}\|v\|_{\widetilde{L}^{q_2}_T(B^s_{p,r})} + \|u\|_{\widetilde{L}^{q_3}_T(B^s_{p,r})}\|v\|_{L^{q_4}_T(L^\infty)}\right),$$
$$\frac{1}{q} := \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}.$$

One also has the estimates for the composition of functions in Besov spaces.

Proposition 3.8. *Let $F : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then for any $s > 0$, $(q, p, r) \in [1, +\infty]^3$, we have*

$$\|\nabla(F(a))\|_{\widetilde{L}^q_T(B^{s-1}_{p,r})} \leq C(F', \|a\|_{L^\infty_T(L^\infty)})\|\nabla a\|_{\widetilde{L}^q_T(B^{s-1}_{p,r})}.$$

If furthermore $F(0) = 0$, then $\|F(a)\|_{\widetilde{L}^q_T(B^s_{p,r})} \leq C(F', \|a\|_{L^\infty_T(L^\infty)})\,\|a\|_{\widetilde{L}^q_T(B^s_{p,r})}$.

In the next section we will need also some notions about *homogeneous paradifferential calculus*: let us recall them. As above, we refer to Chapter 2 of [3] for a detailed presentation: we just point out here that, in the homogeneous setting, the definition of Besov spaces has to be slightly modified, working with the class of distributions $\mathcal{S}'_h$ defined to be the subset of tempered distributions u with $\|\theta(2^{-j}D)u\|_{L^\infty} \to 0$ as $j \to -\infty$ for any $\theta \in \mathcal{D}(\mathbb{R}^d)$ (see Definition 1.26 of [3]).

The homogeneous dyadic blocks $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$ are defined by

$$\dot{\Delta}_j u := \varphi(2^{-j}D)u \qquad \text{if} \quad j \in \mathbb{Z}.$$

The homogeneous low frequency cut-offs are defined by:

$$\dot{S}_j u := \chi(2^{-j}D)u \qquad \text{for} \quad j \in \mathbb{Z}.$$

The homogeneous paraproduct operator and remainder operator are defined by:

$$\dot{T}_u v := \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}u\,\dot{\Delta}_j v \qquad \text{and} \qquad \dot{R}(u, v) := \sum_{j\in\mathbb{Z}}\sum_{|j'-j|\leq 1} \dot{\Delta}_j u\,\dot{\Delta}_{j'} v.$$

Notice that, for all u and v in $\mathcal{S}'_h$, the sequence element $(\dot{S}_{j-1}u\,\dot{\Delta}_j v)$ is spectrally supported in dyadic annuli. The analogous of Proposition 3.5 holds true also in the homogeneous setting.

Let us set $\dot{C}^s = \dot{B}^s_{\infty,\infty}$, for $s > 0$, to be the homogeneous Hölder space. Recall that, for any $u \in \dot{C}^s$, the equality $u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u$ holds. For homogeneous Hölder spaces and time-dependent homogeneous Hölder spaces, we have the following characterization, which will be used for the localization argument in Subsection 4.1.

Proposition 3.9. $\forall \epsilon \in (0,1)$*, there exists a constant* C *such that for all* $u \in \mathcal{S}'_h$*,*

$$C^{-1}\|u\|_{\dot{C}^\epsilon} \le \left\| \frac{\|u(x+y)-u(x)\|_{L^\infty_x}}{|y|^\epsilon} \right\|_{L^\infty_y} \le C\|u\|_{\dot{C}^\epsilon}, \tag{22}$$

and

$$C^{-1}\|u\|_{\widetilde{L}^1_t(\dot{C}^\epsilon)} \le \left\| \int_0^t \frac{\|u(t',x+y)-u(t',x)\|_{L^\infty_x}}{|y|^\epsilon}\, dt' \right\|_{L^\infty_y} \le C\|u\|_{\widetilde{L}^1_t(\dot{C}^\epsilon)}. \tag{23}$$

Proof. The proof of (22) can be found at page 75 of [3].

Let us first show the left-hand inequality of (23). Since

$$\dot{\Delta}_j u(t,x) = 2^{jd}\int_{\mathbb{R}^d} h(2^j y)(u(t,x-y)-u(t,x))\,\mathrm{d}y,$$

where h denotes the inverse Fourier transform of φ, then we easily find

$$\begin{aligned}\int_0^t 2^{j\epsilon}\|\dot{\Delta}_j u(t',\cdot)\|_{L^\infty_x} &\le 2^{jd}\int_{\mathbb{R}^d} 2^{j\epsilon}|y|^\epsilon |h(2^j y)| \int_0^t \frac{\|u(t',x-y)-u(t',x)\|_{L^\infty_x}}{|y|^\epsilon}\,\mathrm{d}t'\,\mathrm{d}y \\ &\le C\left\| \int_0^t \frac{\|u(t',x+y)-u(t',x)\|_{L^\infty_x}}{|y|^\epsilon}\mathrm{d}t' \right\|_{L^\infty_y}.\end{aligned}$$

This relation gives us the left-hand side inequality of (23).

The inverse inequality follows immediately after similar changes with respect to time in the classical proof: let us just sketch it for reader's convenience. Recalling

$$\dot{\Delta}_j = \sum_{|j-j'|\le 1} \dot{\Delta}_j \dot{\Delta}_{j'},$$

we rewrite $\dot{\Delta}_j(u(t,x+y)-u(t,x))$ as follows

$$\begin{aligned}\dot{\Delta}_j(u(t,x+y)-u(t,x)) &= \sum_{|j-j'|\le 1} 2^{jd}\int_{\mathbb{R}^d}\Big(h(2^j(x+y-z)) - h(2^j(x-z))\Big)\dot{\Delta}_{j'}u(t,z)\,\mathrm{d}z \\ &= \sum_{|j-j'|\le 1} 2^{jd}\Big(\int_0^1 (2^j y)\cdot\nabla h(2^j\cdot + 2^j s y)\mathrm{d}s\Big) * \Big(\dot{\Delta}_{j'}u(t,\cdot)\Big).\end{aligned}$$

Therefore there holds

$$\int_0^t \frac{\|\dot\Delta_j(u(t',x+y)-u(t',x))\|_{L^\infty_x}}{|y|^\epsilon}\,\mathrm{d}t' \le C\int_0^t 2^j|y|^{1-\epsilon}\sum_{|j-j'|\le 1}\|\dot\Delta_{j'}u(t')\|_{L^\infty_x}\mathrm{d}t'.$$

On the other side, it is easy to see that, for any $j\in\mathbb{Z}$,

$$\|\dot\Delta_j(u(t',x+y)-u(t',x))\|_{L^\infty_x} \le 2\|\dot\Delta_j u(t')\|_{L^\infty_x}.$$

Let us choose the integer j_y such that $\frac{1}{|y|}\le 2^{j_y}<\frac{2}{|y|}$ and decompose

$$u(t',x+y)-u(t',x)=\sum_{j\le j_y}\dot\Delta_j(u(t',x+y)-u(t',x))+\sum_{j>j_y}\dot\Delta_j(u(t',x+y)-u(t',x)).$$

Hence one arrives at

$$\begin{aligned}
&\int_0^t \frac{\|u(t',x+y)-u(t',x)\|_{L^\infty_x}}{|y|^\epsilon}\,\mathrm{d}t'\\
&\quad\le C\sum_{j\le j_y}\int_0^t 2^j|y|^{1-\epsilon}\sum_{|j-j'|\le 1}\|\dot\Delta_{j'}u(t')\|_{L^\infty_x}\mathrm{d}t' + 2\sum_{j>j_y}\int_0^t \frac{\|\dot\Delta_j u(t')\|_{L^\infty_x}}{|y|^\epsilon}\mathrm{d}t'\\
&\quad= C\sum_{j\le j_y}(2^j|y|)^{1-\epsilon}\int_0^t 2^{j\epsilon}\sum_{|j-j'|\le 1}\|\dot\Delta_{j'}u(t')\|_{L^\infty_x}\mathrm{d}t' + 2\sum_{j>j_y}(2^j y)^{-\epsilon}\int_0^t 2^{j\epsilon}\|\dot\Delta_j u(t')\|_{L^\infty_x}\mathrm{d}t'\\
&\quad\le C\Big(\sum_{j\le j_y}(2^j|y|)^{1-\epsilon}+\sum_{j>j_y}(2^j y)^{-\epsilon}\Big)\sup_{j\in\mathbb{Z}}\int_0^t 2^{j\epsilon}\|\dot\Delta_j u(t')\|_{L^\infty_x}\mathrm{d}t'.
\end{aligned}$$

Noticing that $\epsilon\in(0,1)$ and the choosing of j_y, the right-hand side inequality in (23) easily follows. $\square$

Remark 3.10. From the above proof, it is straightforward to see that the equivalence (23) can be generalized to a function sequence $\{u_n\}_{n\in\mathbb{N}}$ in the following term:

$$\sup_{j\in\mathbb{Z}}\int_0^t \sup_{n\in\mathbb{N}} 2^{j\epsilon}\|\dot\Delta_j(u_n)(t',x)\|_{L^\infty_x}\mathrm{d}t' \sim \left\|\int_0^t \sup_{n\in\mathbb{N}}\frac{\|u_n(t',x+y)-u_n(t',x)\|_{L^\infty_x}}{|y|^\epsilon}\,\mathrm{d}t'\right\|_{L^\infty_y}. \tag{24}$$

Finally let us recall the classical a priori estimates for heat equations in homogeneous Besov spaces (see Section 3.4 of book [3]).

Proposition 3.11. *For any $s \in \mathbb{R}$, there exists a constant C such that*

$$\|\dot{\Delta}_j F(t)\|_{L^\infty} \le C\Big(e^{-t2^{2j}}\|\dot{\Delta}_j f_0\|_{L^\infty} + \int_0^t e^{-(t-t')2^{2j}}\|\dot{\Delta}_j f(t')\|_{L^\infty}\,dt'\Big), \tag{25}$$

where $f_0, f, F \in \mathcal{S}'_h(\mathbb{R}^d)$ and they are linked by the relation

$$F(t,x) = e^{t\Delta} f_0 + \int_0^t e^{(t-t')\Delta} f(t')\,dt'.$$

4. A priori estimates for parabolic and transport equations in endpoint Besov spaces

The present section is devoted to obtain new a priori estimates for parabolic and transport equations in endpoint Besov spaces. These estimates will be the key point to get our results.

In the first subsection we will prove a priori estimates for the parabolic equations in Chemin–Lerner spaces (essentially Hölder in space variable). This will be useful in the proof of Theorem 2.1.

In the second subsection we will prove refined a priori estimates in the endpoint space $B^0_{\infty,1}$ for linear transport equations. This is a generalization of similar results in [21,28], and it will be fundamental in getting lower bounds for the lifespan of the solutions.

4.1. Parabolic estimates in $B^s_{\infty,r}(\mathbb{R}^d)$

The present subsection is devoted to state new a priori estimates for linear parabolic equations with variable coefficients in divergence form (see (26) below), in endpoint Besov spaces $\widetilde{L}^q_T(B^s_{\infty,r})$, which are of independent interest.

Proposition 4.1. *Let ρ be a smooth solution to the following linear parabolic equation*

$$\begin{cases} \partial_t\rho - \operatorname{div}(\kappa\nabla\rho) = f, \\ \rho|_{t=0} = \rho_0, \end{cases} \tag{26}$$

where κ, f, ρ_0 are smooth functions such that

$$0 < \rho_* \le \rho_0(x) \le \rho^*, \quad 0 < \kappa_* \le \kappa(t,x) \le \kappa^*, \quad \forall t \in [0,+\infty), \quad \forall x \in \mathbb{R}^d.$$

Let $s > 0$, $r \in [1,\infty]$ and take any $\epsilon \in (0,1)$, then there exists a constant C (depending on d, s, ϵ, r, ρ_*, ρ^*, κ_*, κ^*) such that the following estimate holds true:

$$\|\rho\|_{\widetilde{L}^\infty_t(B^s_{\infty,r}(\mathbb{R}^d))\cap\widetilde{L}^1_t(B^{s+2}_{\infty,r}(\mathbb{R}^d))} \le C\left[1+\|\kappa\|^{\frac{2}{\epsilon(1-\epsilon)}}_{L^\infty_t(C^\epsilon(\mathbb{R}^d))}\right] \times \Bigg(\|\rho_0\|_{B^s_{\infty,r}(\mathbb{R}^d)} + \|f\|_{\widetilde{L}^1_t(B^s_{\infty,r}(\mathbb{R}^d))}$$

$$+ \int_0^t \Big(\Big[1+\|\kappa\|^{2/(1+\epsilon)}_{C^{1+\epsilon}(\mathbb{R}^d)}\Big]\|\rho\|_{B^s_{\infty,r}(\mathbb{R}^d)}$$

$$+ \|\nabla\kappa\|_{L^\infty(\mathbb{R}^d)} \|\nabla\rho\|_{B^s_{\infty,r}(\mathbb{R}^d)}$$

$$+ \|\nabla\kappa\|_{B^s_{\infty,r}(\mathbb{R}^d)} \|\nabla\rho\|_{L^\infty(\mathbb{R}^d)}\Big) dt'\Bigg). \tag{27}$$

Remark 4.2. By the interpolation-type inequality in Chemin–Lerner space (in the same spirit of (17) and see e.g. the estimate (42) for J_1 (41) on Page 5095 below) and Gronwall's lemma, it is easy to derive the following a priori estimate for ρ from (27):

$$\|\rho\|_{\widetilde{L}^\infty_t(B^s_{\infty,r}(\mathbb{R}^d))\cap\widetilde{L}^1_t(B^{s+2}_{\infty,r}(\mathbb{R}^d))} \leq C_1 \exp\{C_1 K(t)\} \left(\|\rho_0\|_{B^s_{\infty,r}(\mathbb{R}^d)} + \|f\|_{\widetilde{L}^1_t(B^s_{\infty,r}(\mathbb{R}^d))}\right),$$

where $C_1 = C_1(t)$ depends on d, s, ϵ, r, ρ_*, ρ^*, κ_*, κ^* and $\|\kappa\|_{L^\infty_t(C^\epsilon(\mathbb{R}^d))}$, and

$$K(t) := \int_0^t \left(1 + \|\nabla\kappa\|^2_{L^\infty(\mathbb{R}^d)} + \|\nabla\kappa\|^{\max\{2/(1+s),1\}}_{B^s_{\infty,r}(\mathbb{R}^d)}\right) \mathrm{d}t'.$$

Proposition 4.1 will be proved in three steps. The strategy is the following.

First of all, we localize the function ρ into countable functions ϱ_n, each of which is supported on some ball $B(x_n, \delta)$, with small radius $\delta \in (0, 1)$ to be determined in the proof. Hence up to some small perturbation, ϱ_n verifies a heat equation with a time-dependent heat-conduction coefficient (see System (30) below). Consequently, changing the time variable and making use of estimates for the heat equation entail a control for $\|\dot{\Delta}_j\varrho_n(t)\|_{L^\infty}$ for any n. This will be done in Step 1.

In Step 2, thanks to Proposition 3.9 or Remark 3.10, we carry the result from $\{\varrho_n\}$ to ρ, in order to get Hölder type estimates for ρ. Note that Maximum Principle applied to parabolic equations has already given us the control on low frequencies of the solution:

$$\|\rho\|_{L^\infty_t(L^\infty)} \leq \|\rho_0\|_{L^\infty} + \int_0^t \|f(t')\|_{L^\infty}\mathrm{d}t'. \tag{28}$$

Then we have just to exhibit a control for ρ in Hölder space $\widetilde{L}^\infty_t(\dot{C}^\epsilon) \cap \widetilde{L}^1_t(\dot{C}^{2+\epsilon})$ which, thanks to Proposition 3.9, can be bounded locally by (up to a low-frequency control)

$$I(t) := \sup_{j\in\mathbb{Z}} \sup_n 2^{j\epsilon} \|\dot{\Delta}_j\varrho_n\|_{L^\infty_t(L^\infty)} + \sup_{j\in\mathbb{Z}} \int_0^t \sup_n 2^{j(2+\epsilon)} \|\dot{\Delta}_j\varrho_n(t')\|_{L^\infty}\mathrm{d}t'. \tag{29}$$

The result in Step 1 helps us to control $I(t)$, but some complications are due to the fact that we work with a double decomposition, both in space and in frequencies, which do not commute: this will be handled by a careful analysis of each remainder term (coming either from space or frequency localization), combined with a suitable choice of the radius of the balls δ. Finally, let us just mention that the interpolation inequality in Chemin–Lerner space (see e.g. (42) below) helps to control the lower order term.

Step 3 is devoted to the proof of the estimates in general Besov spaces of type $B^s_{\infty,r}$. A localization in Fourier variables of the system, the application of Hölder estimates (established in Step 2) for each dyadic term and a careful calculation on commutators will finally yield the general result.

We agree that in this subsection $\{\varrho_n(t,x)\}$ always denote localized functions of $\rho(t,x)$ in x-*space*, while ρ_j as usual, denotes $\Delta_j\rho$ (localization in the *phase space*).

4.1.1. *Step 1: estimate for* $\dot{\Delta}_j\varrho_n$

Let us take first a smooth partition of unity $\{\psi_n\}_{n\in\mathbb{N}}$ subordinated to a locally finite covering of $\mathbb{R}^d$, depending on some parameter $\delta > 0$. We suppose that the ψ_n's satisfy the following conditions:

(i) $\operatorname{Supp}\psi_n \subset B(x_n,\delta) := B_n$, with $\delta < 1$ to be determined later;
(ii) $\sum_n \psi_n \equiv 1$;
(iii) $0 \le \psi_n \le 1$, with $\psi_n \equiv 1$ on $B(x_n,\delta/2)$;
(iv) $\|\nabla^\eta\psi_n\|_{L^\infty} \le C\,\delta^{-|\eta|}$, for $|\eta| \le 3$;
(v) for each $x \in \mathbb{R}^d$, there are at most N_d (depending on the dimension d) elements in $\{\psi_n\}_{n\in\mathbb{N}}$ covering the ball $B(x,\delta/4)$.

Now by multiplying ψ_n to Equation (26), we get the equation for $\varrho_n := \rho\psi_n$, which is compactly supported on B_n:

$$\begin{cases} \partial_t\varrho_n - \bar{\kappa}_n\Delta\varrho_n = (\kappa - \bar{\kappa}_n)\Delta\varrho_n + g_n, \\ \varrho_n|_{t=0} = \varrho_{0,n} = \psi_n\rho_0, \end{cases} \tag{30}$$

with the time-dependent coefficient

$$\bar{\kappa}_n(t) := \frac{1}{\operatorname{vol}(B_n)}\int_{B_n}\kappa(t,y)\,\mathrm{d}y \in [\kappa_*,\kappa^*],$$

and the remainder term

$$g_n = \nabla\kappa\cdot\nabla\varrho_n - 2\kappa\nabla\psi_n\cdot\nabla\rho - (\kappa\Delta\psi_n + \nabla\kappa\cdot\nabla\psi_n)\rho + f\psi_n. \tag{31}$$

In order to get rid of the variable coefficient $\bar{\kappa}_n(t)$, let us make the one-to-one change in time variable

$$\tau := \tau(t) = \int_0^t \bar{\kappa}_n(t')\,\mathrm{d}t'. \tag{32}$$

Therefore, the new unknown

$$\widetilde{\varrho}_n(\tau,x) := \varrho_n(t,x),$$

satisfies (observe that $\frac{d\tau}{dt} = \bar{\kappa}_n(t)$)

$$\begin{cases} \partial_\tau \widetilde{\varrho}_n - \Delta \widetilde{\varrho}_n = \left(\frac{\widetilde{\kappa}(\tau)}{\widetilde{\kappa}_n(\tau)} - 1 \right) \Delta \widetilde{\varrho}_n + \frac{\widetilde{g}_n(\tau)}{\widetilde{\kappa}_n(\tau)} := \widetilde{G}_n(\tau, x), \\ \widetilde{\varrho}_n|_{\tau=0} = \varrho_{0,n}, \end{cases} \tag{33}$$

where $\widetilde{\kappa}(\tau, x) = \kappa(t, x)$, $\widetilde{\kappa}_n(\tau) = \bar{\kappa}_n(t)$, $\widetilde{\rho}(\tau, x) = \rho(t, x)$, $\widetilde{g}_n(\tau, x) = g_n(t, x)$. We then can apply Proposition 3.11 to the above heat equation to get

$$\|\dot{\Delta}_j \widetilde{\varrho}_n(\tau)\|_{L^\infty} \leq C \Big(\mathrm{e}^{-\tau 2^{2j}} \|\dot{\Delta}_j \varrho_{0,n}\|_{L^\infty} + \int_0^\tau \mathrm{e}^{-(\tau - s)2^{2j}} \|\dot{\Delta}_j \widetilde{G}_n(s)\|_{L^\infty} \, \mathrm{d}s \Big).$$

By virtue of the one-to-one change of time variables (32), we arrive at from the above that

$$\|\dot{\Delta}_j \varrho_n(t)\|_{L^\infty} \leq C \Big(\mathrm{e}^{-\int_0^t \bar{\kappa}_n(t')\mathrm{d}t' 2^{2j}} \|\dot{\Delta}_j \varrho_{0,n}\|_{L^\infty} + \int_0^t \mathrm{e}^{-\int_{t'}^t \bar{\kappa}_n(t'')\mathrm{d}t'' 2^{2j}} \|\dot{\Delta}_j G_n(t')\|_{L^\infty} \bar{\kappa}_n(t') \, \mathrm{d}t' \Big),$$

where G_n is a function supported on B_n, defined by

$$G_n(t, x) = \left(\frac{\kappa(t, x)}{\bar{\kappa}_n(t)} - 1 \right) \Delta \varrho_n(t, x) + \frac{g_n(t, x)}{\bar{\kappa}_n(t)}. \tag{34}$$

Noticing $\bar{\kappa}_n \in [\kappa_*, \kappa^*]$, we actually have

$$\|\dot{\Delta}_j \varrho_n(t)\|_{L^\infty} \leq C \Big(\mathrm{e}^{-\kappa_* 2^{2j} t} \|\dot{\Delta}_j \varrho_{0,n}\|_{L^\infty} + \int_0^t \mathrm{e}^{-\kappa_* 2^{2j}(t - t')} \|\dot{\Delta}_j G_n(t')\|_{L^\infty} \, \mathrm{d}t' \Big). \tag{35}$$

4.1.2. Step 2: Hölder estimates for ρ

Now we come back to consider $\rho = \sum_n \varrho_n$. Thanks to Proposition 3.9 we have

$$\begin{aligned} \|\rho\|_{\widetilde{L}^1_t(\dot{C}^\epsilon)} &\leq C \left\| \int_0^t \frac{\|\rho(t', x + y) - \rho(t', x)\|_{L^\infty_x}}{|y|^\epsilon} \mathrm{d}t' \right\|_{L^\infty_y} \\ &\leq C \sup_{|y| > \delta/4} \int_0^t \frac{\|\rho(t', x + y) - \rho(t', x)\|_{L^\infty_x}}{|y|^\epsilon} \mathrm{d}t' \\ &\quad + C \sup_{|y| \leq \delta/4} \int_0^t \frac{\|\rho(t', x + y) - \rho(t', x)\|_{L^\infty_x}}{|y|^\epsilon} \mathrm{d}t', \end{aligned}$$

whose second term can be controlled by (noticing the assumption (v) on the partition of unity $\{\psi_n\}$)

$$N_d \, C \sup_{|y - z| \leq \delta/4} \int_0^t \sup_n \frac{\|\varrho_n(t', x + y) - \varrho_n(t', x + z)\|_{L^\infty_x}}{|y - z|^\epsilon} \mathrm{d}t'.$$

Thus, keeping in mind Remark 3.10, we find that

$$\|\rho\|_{\widetilde{L}^1_t(\dot{C}^\epsilon)} \leq C\delta^{-\epsilon}\int_0^t \|\rho\|_{L^\infty}\mathrm{d}t' + N_d C \sup_{j\in\mathbb{Z}}\int_0^t \sup_n 2^{j\epsilon}\|\dot{\Delta}_j\varrho_n(t')\|_{L^\infty}\mathrm{d}t'.$$

Similarly (with L^1 replaced by L^∞ in the time variable), we have

$$\|\rho\|_{\widetilde{L}^\infty_t(\dot{C}^\epsilon)} \leq C\delta^{-\epsilon}\|\rho\|_{L^\infty_t(L^\infty)} + C\sup_{j\in\mathbb{Z}}\sup_n 2^{j\epsilon}\|\dot{\Delta}_j\varrho_n\|_{L^\infty_t(L^\infty)}.$$

Since $\nabla^2\rho = \sum_n(\nabla^2\varrho_n)$, from the same arguments as before we infer

$$\|\rho\|_{\widetilde{L}^1_t(\dot{C}^{2+\epsilon})} \leq C\|\nabla^2\rho\|_{\widetilde{L}^1_t(\dot{C}^\epsilon)} \leq C\delta^{-\epsilon}\int_0^t \|\nabla^2\rho\|_{L^\infty}\mathrm{d}t' + C\sup_{j\in\mathbb{Z}}\int_0^t \sup_n 2^{j\epsilon}\|\dot{\Delta}_j\nabla^2\varrho_n(t')\|_{L^\infty}\mathrm{d}t'.$$

Thus we get

$$\|\rho\|_{\widetilde{L}^\infty_t(\dot{C}^\epsilon)\cap\widetilde{L}^1_t(\dot{C}^{2+\epsilon})} \leq C\delta^{-\epsilon}\Big(\|\rho\|_{L^\infty_t(L^\infty)} + \int_0^t \|\nabla^2\rho\|_{L^\infty}\mathrm{d}t'\Big) + CI(t), \tag{36}$$

with $I(t)$ defined in (29).

We next control $I(t)$: recalling the estimates for $\|\dot{\Delta}_j\varrho_n(t)\|_{L^\infty}$ in (35), we know that

$$\begin{aligned} I(t) \leq\ & C\sup_{j\in\mathbb{Z}}\sup_n\Big(2^{j\epsilon}\|\dot{\Delta}_j\varrho_{0,n}\|_{L^\infty} + 2^{j\epsilon}\int_0^t \|\dot{\Delta}_j G_n(t')\|_{L^\infty}\mathrm{d}t'\Big) \\ & + C\sup_{j\in\mathbb{Z}}\int_0^t \sup_n\Big(2^{j(2+\epsilon)}e^{-\kappa_* 2^{2j}t'}\|\dot{\Delta}_j\varrho_{0,n}\|_{L^\infty} \\ & + \int_0^{t'} 2^{j(2+\epsilon)}e^{-\kappa_* 2^{2j}(t'-t'')}\|\dot{\Delta}_j G_n(t'')\|_{L^\infty}\mathrm{d}t''\Big)\mathrm{d}t'. \end{aligned}$$

Noticing $\int_0^t 2^{2j}e^{-\kappa_* 2^{2j}t'}\mathrm{d}t' \leq 1/\kappa_*$, $I(t)$ can be bounded as (by Young's inequality)

$$\begin{aligned} I(t) &\leq C\sup_{j\in\mathbb{Z}}\sup_n 2^{j\epsilon}\|\dot{\Delta}_j\varrho_{0,n}\|_{L^\infty} + C\sup_{j\in\mathbb{Z}}\int_0^t \sup_n 2^{j\epsilon}\|\dot{\Delta}_j G_n(t')\|_{L^\infty}\mathrm{d}t' \\ &\leq C\sup_n\|\varrho_{0,n}\|_{\dot{C}^\epsilon} + C\sup_j\int_0^t \sup_n 2^{j\epsilon}\|\dot{\Delta}_j G_n(t')\|_{L^\infty}\mathrm{d}t'. \end{aligned}$$

Recalling the definition (34) of G_n, we next control the second term on the righthand side by decomposing it into two parts I_i, $i = 1, 2$:

$$I_1(t) := \sup_j \int_0^t \sup_n 2^{j\epsilon} \left\| \dot{\Delta}_j \left(\Big(\frac{\kappa(t', x)}{\bar{\kappa}_n(t')} - 1 \Big) \Delta \varrho_n(t', x) \right) \right\|_{L^\infty_x} \mathrm{d}t',$$

$$I_2(t) := \sup_j \int_0^t \sup_n 2^{j\epsilon} \left\| \dot{\Delta}_j \left(\frac{g_n(t', x)}{\bar{\kappa}_n(t')} \right) \right\|_{L^\infty_x} \mathrm{d}t'.$$

From now on let us fix some positive time $t_0 \in \mathbb{R}^+$ and we will always consider in the sequel on the time interval $[0, t_0]$.

Firstly, by Proposition 3.9, there exists a positive constant C such that

$$|\kappa(t, x) - \kappa(t, y)| \le C \|\kappa\|_{L^\infty_{t_0}(\dot{C}^\epsilon)} |x - y|^\epsilon, \quad \forall x, y \in \mathbb{R}^d,\ t \in [0, t_0]. \tag{37}$$

Notice $\bar{\kappa}_n \ge \kappa_* > 0$, which ensures that, for all $t \in [0, t_0]$,

$$\|\kappa/\bar{\kappa}_n - 1\|_{L^\infty(B_n)} \le \kappa_*^{-1} \left\| \frac{1}{\mathrm{vol}(B_n)} \int_{B_n} \big(\kappa(t, x) - \kappa(t, y)\big)\, \mathrm{d}y \right\|_{L^\infty(B_n)} \le C \|\kappa\|_{L^\infty_{t_0}(\dot{C}^\epsilon)} \kappa_*^{-1} \delta^\epsilon. \tag{38}$$

Then, by Bony's decomposition for products, it is straightforward to deduce the following estimates by Proposition 3.5:

$$\begin{aligned}
&2^{j\epsilon} \left\| \dot{\Delta}_j \left(\Big(\frac{\kappa(t', x)}{\bar{\kappa}_n(t')} - 1 \Big) \Delta \varrho_n(t', x) \right) \right\|_{L^\infty_x} \\
&\le C 2^{j\epsilon} \sum_{|j'-j| \le 3} \left\| \kappa(t', x)/\bar{\kappa}_n(t') - 1 \right\|_{L^\infty(B_n)} \|\dot{\Delta}_{j'} \Delta \varrho_n(t')\|_{L^\infty} \\
&\quad + C 2^{j\epsilon} \sum_{|j'-j| \le 3} \left\| \dot{\Delta}_{j'} \Big(\kappa(t', x)/\bar{\kappa}_n(t') - 1 \Big) \right\|_{L^\infty} \|\Delta \varrho_n(t')\|_{L^\infty} \\
&\quad + C \sum_{j' \ge j-3} 2^{-(j'-j)\epsilon} 2^{j'\epsilon} \left\| \dot{\Delta}_{j'} \Big(\kappa(t', x)/\bar{\kappa}_n(t') - 1 \Big) \right\|_{L^\infty} \|\Delta \varrho_n(t')\|_{L^\infty}.
\end{aligned}$$

By view of Inequality (38), we have correspondingly the following bound for $I_1(t)$, $t \in [0, t_0]$,

$$\begin{aligned}
I_1(t) \le C \|\kappa\|_{L^\infty_{t_0}(\dot{C}^\epsilon)} \delta^\epsilon \underbrace{\sup_j \int_0^t \sup_n 2^{j\epsilon} \|\dot{\Delta}_j \Delta \varrho_n(t')\|_{L^\infty} \mathrm{d}t'}_{\le I(t)} \\
+ C \int_0^t \left\| \kappa(t', x) \right\|_{\dot{C}^\epsilon} \sup_n \|\Delta \varrho_n(t')\|_{L^\infty} \, \mathrm{d}t'.
\end{aligned}$$

Recalling the definition of $\varrho_n := \varrho\,\psi_n$, it is easy to see that

$$\begin{aligned}\|\Delta\varrho_n(t')\|_{L^\infty} &\leq \|\Delta\psi_n\|_{L^\infty}\|\rho(t')\|_{L^\infty} + 2\|\nabla\psi_n\|_{L^\infty}\|\nabla\rho(t')\|_{L^\infty} + \|\Delta\rho(t')\|_{L^\infty}\\ &\leq C\delta^{-2}\|\rho(t')\|_{L^\infty} + C\delta^{-1}\|\nabla\rho(t')\|_{L^\infty} + \|\Delta\rho(t')\|_{L^\infty}.\end{aligned}$$

Thus $I_1(t)$ is bounded as (noticing $\delta^{-1}\|\nabla\rho(t')\|_{L^\infty} \leq C(\delta^{-2}\|\rho(t')\|_{L^\infty} + \|\Delta\rho(t')\|_{L^\infty})$)

$$I_1(t) \leq C\|\kappa\|_{L^\infty_{t_0}(\dot{C}^\epsilon)}\delta^\epsilon\, I(t) + C\int_0^t \|\kappa(t',x)\|_{\dot{C}^\epsilon}(\delta^{-2}\|\rho(t')\|_{L^\infty} + \|\Delta\rho(t')\|_{L^\infty})\mathrm{d}t'.$$

Similarly, we get by Proposition 3.5 (recalling the definition (31) for g_n)

$$\begin{aligned}I_2(t) &\leq C\sup_j\int_0^t \sup_n 2^{j\epsilon}\|\dot{\Delta}_j g_n(t')\|_{L^\infty}\mathrm{d}t'\\ &\leq C\int_0^t \Big(\|\kappa(t')\|_{\dot{C}^\epsilon}(\delta^{-1}\|\nabla\rho(t')\|_{L^\infty} + \delta^{-2}\|\rho(t')\|_{L^\infty})\\ &\quad + \|\nabla\kappa(t')\|_{\dot{C}^\epsilon}(\delta^{-1}\|\rho(t')\|_{L^\infty} + \|\nabla\rho(t')\|_{L^\infty})\\ &\quad + \|\nabla\kappa(t')\|_{L^\infty}(\|\rho(t')\|_{\dot{C}^{1+\epsilon}} + \delta^{-1}\|\rho(t')\|_{\dot{C}^\epsilon} + \delta^{-1-\epsilon}\|\rho(t')\|_{L^\infty})\\ &\quad + \big(\delta^{-1}\|\nabla\rho(t')\|_{\dot{C}^\epsilon} + \delta^{-2-\epsilon}\|\rho(t')\|_{L^\infty}\big)\Big)\mathrm{d}t' + C(\delta^{-\epsilon}+1)\|f\|_{\widetilde{L}^1_t(C^\epsilon)}.\end{aligned}$$

To conclude, we can choose

$$\delta^{-\epsilon} = 1 + \widetilde{C}\|\kappa\|_{L^\infty_{t_0}(\dot{C}^\epsilon)},\ \text{ with } \widetilde{C} \text{ depending only on } d,$$

such that for all $t \in [0, t_0]$

$$\begin{aligned}I(t) &\leq C\,\delta^{-\epsilon}(\|\rho_0\|_{C^\epsilon} + \|f\|_{\widetilde{L}^1_t(C^\epsilon)}) + C\int_0^t (\delta^{-1}\|\nabla\kappa(t')\|_{\dot{C}^\epsilon} + \delta^{-2-\epsilon})\|\rho(t')\|_{L^\infty}\mathrm{d}t'\\ &\quad + C\int_0^t \Big(\|\kappa(t')\|_{\dot{C}^\epsilon}\|\Delta\rho(t')\|_{L^\infty} + (\|\nabla\kappa(t')\|_{\dot{C}^\epsilon} + \delta^{-1}\|\kappa(t')\|_{\dot{C}^\epsilon})\|\nabla\rho(t')\|_{L^\infty}\\ &\quad + (\|\nabla\kappa(t')\|_{L^\infty} + \delta^{-1})\|\rho(t')\|_{\dot{C}^{1+\epsilon}} + \delta^{-1}\|\nabla\kappa(t')\|_{L^\infty}\|\rho(t')\|_{\dot{C}^\epsilon}\Big)\mathrm{d}t',\end{aligned}$$

where we have noticed the following by interpolation inequality

$$\delta^{-2}\|\kappa(t')\|_{\dot{C}^\epsilon},\quad \delta^{-1-\epsilon}\|\nabla\kappa(t')\|_{L^\infty} \lesssim \delta^{-2-\epsilon} + \delta^{-1}\|\nabla\kappa(t')\|_{\dot{C}^\epsilon}.$$

Hence, recalling the estimates (36) for $\|\rho\|_{L^\infty(\dot{C}^\epsilon)\cap\widetilde{L}^1_t(\dot{C}^{2+\epsilon})}$ and (28) for $\|\rho\|_{L^\infty_t(L^\infty)}$, noticing also that $\|f\|_{L^1_t(L^\infty)} \lesssim \|f\|_{\widetilde{L}^1_t(C^\epsilon)}$, one arrives at for all $t \in [0, t_0]$,

$$\begin{aligned}\|\rho\|_{L^\infty_t(C^\epsilon)\cap\widetilde{L}^1_t(C^{2+\epsilon})} &\le C\delta^{-\epsilon}\Big(\|\rho\|_{L^\infty_t(L^\infty)} + \int_0^t (\|\rho(t')\|_{L^\infty} + \|\nabla^2\rho(t')\|_{L^\infty})\mathrm{d}t'\Big) + CI \\ &\le C\,\delta^{-\epsilon}\Big(\|\rho_0\|_{C^\epsilon} + \|f\|_{\widetilde{L}^1_t(C^\epsilon)}\Big) + C\int_0^t K_1(t')\|\rho(t')\|_{L^\infty}\mathrm{d}t' + CJ,\end{aligned} \tag{39}$$

with (recall that $\|\kappa(t')\|_{\dot{C}^\epsilon} \sim \delta^{-\epsilon}$)

$$\begin{aligned}K_1(t') &:= \delta^{-2-\epsilon} + \delta^{-1}\|\kappa(t')\|_{\dot{C}^{1+\epsilon}}, \\ J &:= \int_0^t \Big(\delta^{-\epsilon}\|\nabla^2\rho(t')\|_{L^\infty} + (\|\nabla\kappa(t')\|_{\dot{C}^\epsilon} + \delta^{-1-\epsilon})\|\nabla\rho(t')\|_{L^\infty} \\ &\quad + (\|\nabla\kappa(t')\|_{L^\infty} + \delta^{-1})\|\rho(t')\|_{\dot{C}^{1+\epsilon}} + \delta^{-1}\|\nabla\kappa(t')\|_{L^\infty}\|\rho(t')\|_{\dot{C}^\epsilon}\Big)\mathrm{d}t'.\end{aligned} \tag{40}$$

Next we decompose J into four parts $J_1 \dots J_4$ and we bound them one by one by interpolation-type inequalities (see e.g. (42) for J_1 (41) below). Denote

$$J_1 := \int_0^t \delta^{-\epsilon}\|\nabla^2\rho(t')\|_{L^\infty}\mathrm{d}t'. \tag{41}$$

Then for any $j_0 \in \mathbb{N}$, we can control J_1 as follows:

$$\begin{aligned}J_1 &\le C\int_0^t \delta^{-\epsilon}\sum_j 2^{2j}\|\dot{\Delta}_j\rho(t')\|_{L^\infty}\mathrm{d}t' \\ &\le C\int_0^t \delta^{-\epsilon}\sum_{j\le j_0} 2^{j(2-\epsilon)}2^{j\epsilon}\|\dot{\Delta}_j\rho(t')\|_{L^\infty}\mathrm{d}t' + C\int_0^t \delta^{-\epsilon}\sum_{j>j_0} 2^{-j\epsilon}2^{j(2+\epsilon)}\|\dot{\Delta}_j\rho(t')\|_{L^\infty}\mathrm{d}t' \\ &\le C\int_0^t \delta^{-\epsilon}2^{j_0(2-\epsilon)}\sup_{j\le j_0} 2^{j\epsilon}\|\dot{\Delta}_j\rho(t')\|_{L^\infty}\mathrm{d}t' \\ &\quad + C\int_0^t \delta^{-\epsilon}\sum_{j>j_0} 2^{-(j-j_0)\epsilon}2^{-j_0\epsilon+j(2+\epsilon)}\|\dot{\Delta}_j\rho(t')\|_{L^\infty}\mathrm{d}t'\end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^t \delta^{-\epsilon}2^{j_0(2-\epsilon)}\sup_j 2^{j\epsilon}\|\dot\Delta_j\rho(t')\|_{L^\infty}\mathrm{d}t' \\
&\quad + C\int_0^t \delta^{-\epsilon}\sum_{q>0}2^{-q\epsilon}2^{-j_0\epsilon+(q+j_0)(2+\epsilon)}\|\dot\Delta_{q+j_0}\rho(t')\|_{L^\infty}\mathrm{d}t'.
\end{aligned}
$$

The second term on the righthand side can be bounded as

$$
\begin{aligned}
& C\sum_{q>0}2^{-q\epsilon}\int_0^t \delta^{-\epsilon}2^{-j_0\epsilon+(q+j_0)(2+\epsilon)}\|\dot\Delta_{q+j_0}\rho(t')\|_{L^\infty}\mathrm{d}t' \\
&\quad \le C\sum_{q>0}2^{-q\epsilon}\sup_{q'}\int_0^t \delta^{-\epsilon}2^{-j_0\epsilon+q'(2+\epsilon)}\|\dot\Delta_{q'}\rho(t')\|_{L^\infty}\mathrm{d}t' \\
&\quad \le C_\epsilon\sup_{q'}\int_0^t \delta^{-\epsilon}2^{-j_0\epsilon}2^{q'(2+\epsilon)}\|\dot\Delta_{q'}\rho(t')\|_{L^\infty}\mathrm{d}t',
\end{aligned}
$$

for a suitable constant C_ϵ. For any $\eta>0$ small, let us choose j_0 large enough (namely, $C_\epsilon\,\delta^{-\epsilon}\,2^{-j_0\epsilon}\sim\eta$) such that

$$
J_1 \le C_{\eta,\epsilon}\int_0^t \delta^{-2}\|\rho(t')\|_{\dot C^\epsilon}\mathrm{d}t' + \eta\|\rho\|_{\widetilde L^1_t(\dot C^{2+\epsilon})}. \tag{42}
$$

Along the lines of the proof of the interpolation-type inequality (42) for J_1, we have the bounds for J_2, J_3

$$
\begin{aligned}
J_2 &:= \int_0^t \left(\|\kappa(t')\|_{\dot C^{1+\epsilon}}+\delta^{-1-\epsilon}\right)\|\nabla\rho(t')\|_{L^\infty}\mathrm{d}t' \\
&\le C\int_0^t \left(\|\kappa(t')\|_{\dot C^{1+\epsilon}}+\delta^{-1-\epsilon}\right)^{\frac{2}{1+\epsilon}}\|\rho(t')\|_{\dot C^\epsilon} + \eta\|\rho\|_{\widetilde L^1_t(\dot C^{2+\epsilon})} \\
J_3 &:= \int_0^t \left(\|\nabla\kappa(t')\|_{L^\infty}+\delta^{-1}\right)\|\rho(t')\|_{\dot C^{1+\epsilon}}\mathrm{d}t' \\
&\le C\int_0^t \left(\|\nabla\kappa(t')\|_{L^\infty}+\delta^{-1}\right)^2\|\rho(t')\|_{\dot C^\epsilon} + \eta\|\rho\|_{\widetilde L^1_t(\dot C^{2+\epsilon})}.
\end{aligned}
$$

Finally, we leave J_4 unchanged,

$$J_4 := \int_0^t \delta^{-1} \|\nabla\kappa(t')\|_{L^\infty} \|\rho(t')\|_{\dot{C}^\epsilon} \,\mathrm{d}t'.$$

Therefore, to sum up, choosing η sufficiently small, denoting

$$K_2(t') := \delta^{-2} + \|\kappa(t')\|_{\dot{C}^{1+\epsilon}}^{\frac{2}{1+\epsilon}}, \tag{43}$$

and noticing that (by the choice of δ and interpolation inequalities, e.g. $\|\kappa(t')\|_{\dot{C}^\epsilon} \lesssim \|\kappa(t')\|_{L^\infty}^{\frac{1}{1+\epsilon}} \|\kappa(t')\|_{\dot{C}^{1+\epsilon}}^{\frac{\epsilon}{1+\epsilon}}$)

$$K_2(t') \geq C^{-1}(\delta^{-2} + \|\nabla\kappa(t')\|_{L^\infty}^2 + \delta^{-1}\|\nabla\kappa(t')\|_{L^\infty}),$$

for all $t \in [0, t_0]$, the estimate (39) for ρ becomes

$$\|\rho\|_{\widetilde{L}_t^\infty(C^\epsilon)\cap\widetilde{L}_t^1(C^{2+\epsilon})} \leq C\delta^{-\epsilon}\left(\|\rho_0\|_{C^\epsilon} + \|f\|_{\widetilde{L}_t^1(C^\epsilon)}\right) + C\int_0^t \left(K_1(t')\|\rho(t')\|_{L^\infty} + K_2(t')\|\rho(t')\|_{\dot{C}^\epsilon}\right)\mathrm{d}t'. \tag{44}$$

Notice that recalling the definition of K_1 (40) and K_2 (43), the above estimate (44) with $t = t_0$ entails immediately the conclusion (27) when $s = \epsilon \in (0, 1)$, $r = \infty$, since t_0 can be chosen arbitrarily.

4.1.3. Step 3: general case $B^s_{\infty,r}$

Now we want to deal with the general case $B^s_{\infty,r}$. Let us apply $\widetilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$, $j \geq 0$, to System (26), yielding

$$\begin{cases} \partial_t\overline{\rho}_j - \operatorname{div}(\kappa\nabla\overline{\rho}_j) = \overline{f}_j - \overline{R}_j, \\ \overline{\rho}_j|_{t=0} = \overline{\rho}_{0,j}, \end{cases} \tag{45}$$

with

$$\overline{\rho}_j = \widetilde{\Delta}_j\rho, \quad \overline{f}_j = \widetilde{\Delta}_j f, \quad \overline{R}_j = \operatorname{div}([\kappa, \widetilde{\Delta}_j]\nabla\rho), \quad \overline{\rho}_{0,j} = \widetilde{\Delta}_j\rho_0.$$

We apply the a priori estimate (44) to the solution $\overline{\rho}_j$ of System (45) for some $\epsilon \in (0, 1)$, entailing for all $t \in [0, t_0]$,

$$\|\overline{\rho}_j\|_{\widetilde{L}_t^\infty(C^\epsilon)\cap\widetilde{L}_t^1(C^{2+\epsilon})} \leq C\delta^{-\epsilon}\left(\|\overline{\rho}_{0,j}\|_{C^\epsilon} + \|\overline{f}_j - \overline{R}_j\|_{\widetilde{L}_t^1(C^\epsilon)}\right) + C\int_0^t \left(K_1\|\overline{\rho}_j\|_{L^\infty} + K_2\|\overline{\rho}_j\|_{\dot{C}^\epsilon}\right)\mathrm{d}t', \tag{46}$$

with $\delta^{-\epsilon} = 1 + \widetilde{C}\|\kappa\|_{L^\infty_{t_0}(\dot{C}^\epsilon)}$ and K_1, K_2 defined in (40) and (43) respectively.

Let us notice that for $j \geq 0$, denoting by $\rho_j = \Delta_j \rho$ and $\rho_q = \Delta_q \rho$ as usual, we have

$$\Delta_j \overline{\rho}_j = \rho_j \quad \text{and} \quad \Delta_q \overline{\rho}_j \equiv 0 \text{ if } |q-j| \geq 4.$$

Hence, due to the dyadic characterization of Hölder spaces, the above inequality (46) gives

$$\begin{aligned} & 2^{j\epsilon} \|\rho_j\|_{L^\infty_t(L^\infty)} + 2^{j(2+\epsilon)} \int_0^t \|\rho_j\|_{L^\infty} \mathrm{d}t' \\ & \leq C\delta^{-\epsilon} \left(2^{j\epsilon} \sum_{|j-q|\leq 3} \left(\|\rho_{0,q}\|_{L^\infty} + \int_0^t \|f_q\|_{L^\infty} \mathrm{d}t' \right) + \|\overline{R}_j\|_{\widetilde{L}^1_t(C^\epsilon)} \right) \\ & \quad + C \sum_{|j-q|\leq 3} \int_0^t \left(K_1 \|\rho_q\|_{L^\infty} + 2^{j\epsilon} K_2 \|\rho_q\|_{L^\infty} \right) \mathrm{d}t'. \end{aligned} \tag{47}$$

Let us consider $\|\overline{R}_j\|_{\widetilde{L}^1_t(C^\epsilon)}$ for a while: we decompose the commutator $\overline{R}_j$ into the following five members (with $\widetilde{\kappa} = \kappa - \Delta_{-1}\kappa$):

$$\begin{aligned} & \operatorname{div}([T_{\widetilde{\kappa}}, \widetilde{\Delta_j}]\nabla\rho) + \operatorname{div}(T'_{\widetilde{\Delta}_j \nabla\rho} \widetilde{\kappa}) - \operatorname{div} \widetilde{\Delta_j}(T_{\nabla\rho}\widetilde{\kappa}) - \operatorname{div} \widetilde{\Delta_j}(R(\widetilde{\kappa}, \nabla\rho)) + \operatorname{div}([\Delta_{-1}\kappa, \widetilde{\Delta_j}]\nabla\rho) \\ & \quad := \overline{R}^1_j + \overline{R}^2_j + \overline{R}^3_j + \overline{R}^4_j + \overline{R}^5_j, \ \text{with } T'_u v := T_u v + R(u,v). \end{aligned}$$

It is easy to see that the members $\overline{R}^1_j$, $\overline{R}^3_j$, $\overline{R}^4_j$, $\overline{R}^5_j$ are spectrally supported in the ball of size 2^j and hence

$$\|(\overline{R}^1_j, \overline{R}^3_j, \overline{R}^4_j, \overline{R}^5_j)\|_{\widetilde{L}^1_t(C^\epsilon)} \lesssim 2^{j\epsilon} \int_0^t \|(\overline{R}^1_j, \overline{R}^3_j, \overline{R}^4_j, \overline{R}^5_j)\|_{L^\infty} \, \mathrm{d}t'.$$

By use also of classical commutator estimates Proposition 3.4 on the righthand side one derives (for some suitable sequence $(c_j)_j \in \ell^r$)

$$\|(\overline{R}^1_j, \overline{R}^3_j, \overline{R}^4_j, \overline{R}^5_j)\|_{\widetilde{L}^1_t(C^\epsilon)} \lesssim 2^{j(\epsilon-s)} c_j \int_0^t \left(\|\nabla\kappa\|_{B^s_{\infty,r}} \|\nabla\rho\|_{L^\infty} + \|\nabla\kappa\|_{L^\infty} \|\nabla\rho\|_{B^s_{\infty,r}} \right) \mathrm{d}t'. \tag{48}$$

As for $\overline{R}^2_j$, we control its $\widetilde{L}^1_t(C^\epsilon)$-norm by

$$\begin{aligned} & \sup_{j''\leq j'+3} 2^{j''\epsilon} \int_0^t \left\| \operatorname{div}\left(\Delta_{j''} \sum_{j'\geq j-3} (\Delta_{j'}\widetilde{\kappa} S_{j'+1} \widetilde{\Delta}_j \nabla\rho) \right) \right\|_{L^\infty} \mathrm{d}t' \\ & \lesssim \sup_{j''\leq j'+3} 2^{j''(\epsilon+1)} \int_0^t \sum_{j'\geq j-3} \|\Delta_{j'}\widetilde{\kappa}\|_{L^\infty} \|\widetilde{\Delta}_j \nabla\rho\|_{L^\infty} \mathrm{d}t' \end{aligned}$$

$$\lesssim 2^{j'(\epsilon+1)} \int_0^t \sum_{j' \geq j-3} 2^{-j'(s+1)} c_{j'} \|\nabla\kappa\|_{B^s_{\infty,r}} \|\nabla\rho\|_{L^\infty} \mathrm{d}t'$$

$$\lesssim 2^{j(\epsilon-s)} \sum_{j' \geq j-3} 2^{(j'-j)(\epsilon-s)} c_{j'} \int_0^t \|\nabla\kappa\|_{B^s_{\infty,r}} \|\nabla\rho\|_{L^\infty} \mathrm{d}t'.$$

Therefore we choose $\epsilon < s$ to obtain

$$\|\overline{R}^2_j\|_{\widetilde{L}^1_t(C^\epsilon)} \lesssim 2^{j(\epsilon-s)} c_j \int_0^t \|\nabla\kappa\|_{B^s_{\infty,r}} \|\nabla\rho\|_{L^\infty} \mathrm{d}t'. \tag{49}$$

Finally, by view of the above estimates (48) and (49) for $\overline{R}_j$ term, one gets from a priori estimate (47) the following bound for ρ_j:

$$\begin{aligned}
&\|\rho_j\|_{L^\infty_t(L^\infty)} + 2^{2j} \int_0^t \|\rho_j\|_{L^\infty} \\
&\leq C\delta^{-\epsilon} \Bigg(\sum_{|j-q|\leq 3} \Big(\|\rho_{0,q}\|_{L^\infty} + \int_0^t \|f_q\|_{L^\infty} \Big) \\
&\quad + 2^{-js} c_j \int_0^t \Big(\|\nabla\kappa\|_{B^s_{\infty,r}} \|\nabla\rho\|_{L^\infty} + \|\nabla\kappa\|_{L^\infty} \|\nabla\rho\|_{B^s_{\infty,r}} \Big) \mathrm{d}t' \Bigg) \\
&\quad + C \sum_{|j-q|\leq 3} \int_0^t \Big(2^{-j\epsilon} K_1 \|\rho_q\|_{L^\infty} + K_2 \|\rho_q\|_{L^\infty} \Big) \mathrm{d}t'.
\end{aligned}$$

Therefore, we multiply both sides by 2^{js}, $j \geq -1$ and then take ℓ^r norm, and we arrive at the following: for all $t \in [0, t_0]$ (noticing the L^∞-estimate (28))

$$\begin{aligned}
\|\rho\|_{\widetilde{L}^\infty_t(B^s_{\infty,r}) \cap \widetilde{L}^1_t(B^{s+2}_{\infty,r})} &\leq C \int_0^t (K_1 + K_2) \|\rho\|_{B^s_{\infty,r}} \mathrm{d}t' \\
&\quad + C\delta^{-\epsilon} \Bigg(\|\rho_0\|_{B^s_{\infty,r}} + \|f\|_{\widetilde{L}^1_t(B^s_{\infty,r})} \\
&\quad + \int_0^t \Big(\|\nabla\kappa\|_{L^\infty} \|\nabla\rho\|_{B^s_{\infty,r}} + \|\nabla\kappa\|_{B^s_{\infty,r}} \|\nabla\rho\|_{L^\infty} \Big) \mathrm{d}t' \Bigg).
\end{aligned}$$

Recalling the definitions (40), (43) of K_1, K_2 and $\delta^{-\epsilon} \sim 1 + \|\kappa\|_{L^\infty_{t_0}(\dot{C}^\epsilon)}$, by Young's inequality we infer that

$$\int_0^t (K_1+K_2)\|\rho\|_{B^s_{\infty,r}}\mathrm{d}t' \le C\int_0^t \left(\delta^{-\frac{2}{1-\epsilon}} + \|\kappa(t')\|^{\frac{2}{1+\epsilon}}_{\dot{C}^{1+\epsilon}}\right)\|\rho\|_{B^s_{\infty,r}}\mathrm{d}t',$$

and hence, for all $t\in[0,t_0]$, there holds

$$\begin{aligned}\|\rho\|_{\widetilde{L}^\infty_t(B^s_{\infty,r})\cap\widetilde{L}^1_t(B^{s+2}_{\infty,r})} \le{}& C\left(1+\|\kappa\|^{\frac{2}{\epsilon(1-\epsilon)}}_{L^\infty_{t_0}(\dot{C}^\epsilon)}\right)\Bigg(\|\rho_0\|_{B^s_{\infty,r}} + \|f\|_{\widetilde{L}^1_t(B^s_{\infty,r})}\\ &+\int_0^t\Big(\big(1+\|\kappa(t')\|^{\frac{2}{1+\epsilon}}_{\dot{C}^{1+\epsilon}}\big)\|\rho\|_{B^s_{\infty,r}} + \|\nabla\kappa\|_{L^\infty}\|\nabla\rho\|_{B^s_{\infty,r}}\\ &+\|\nabla\kappa\|_{B^s_{\infty,r}}\|\nabla\rho\|_{L^\infty}\Big)\mathrm{d}t'\Bigg).\end{aligned}$$

By choosing $t=t_0$ above we finished the proof of Proposition 4.1, since $t_0\in\mathbb{R}^+$ can be chosen arbitrarily.

4.2. *Transport equations in* $B^0_{\infty,1}$

We state and prove here refined a priori estimates for transport equations

$$\begin{cases}\partial_t\omega + v\cdot\nabla\omega = g,\\ \omega|_{t=0}=\omega_0,\end{cases}\tag{50}$$

in the endpoint Besov space $B^0_{\infty,1}$.

First of all, let us recall the classical result in the setting of $B^s_{\infty,r}$ classes (see e.g. [3], Chapter 3).

Proposition 4.3. *Let $1\le r\le\infty$ and $s>0$ ($s>-1$ if $\operatorname{div} v=0$). Let $\omega_0\in B^s_{\infty,r}$, $g\in L^1([0,T];B^s_{\infty,r})$ and v be a time-dependent vector field in $\mathcal{C}_b([0,T]\times\mathbb{R}^N)$ such that*

$$\begin{aligned}&\nabla v\in L^1([0,T];L^\infty) &&\text{if } s<1,\\ &\nabla v\in L^1([0,T];B^{s-1}_{\infty,r}) &&\text{if } s>1, \quad\text{or } s=r=1.\end{aligned}$$

Then Equation (50) has a unique solution ω in:

- *the space $\mathcal{C}([0,T];B^s_{\infty,r})$ if $r<\infty$,*
- *the space $\left(\bigcap_{s'<s}\mathcal{C}([0,T];B^{s'}_{\infty,\infty})\right)\bigcap\mathcal{C}_w([0,T];B^s_{\infty,\infty})$ if $r=\infty$.*

Moreover, for all $t\in[0,T]$, we have

$$e^{-CV(t)}\|\omega(t)\|_{B^s_{\infty,r}} \le \|\omega_0\|_{B^s_{\infty,r}} + \int_0^t e^{-CV(t')}\|g(t')\|_{B^s_{\infty,r}}\,dt'\tag{51}$$

with $V'(t) := \begin{cases} \|\nabla v(t)\|_{L^\infty} & \text{if } s < 1, \\ \|\nabla v(t)\|_{B^{s-1}_{\infty,r}} & \text{if } s > 1 \end{cases}$, *or* $s = r = 1$.

If $\omega = v$ *then, for all* $s > 0$ ($s > -1$ *if* $\operatorname{div} v = 0$)*, Estimate* (51) *holds with* $V'(t) := \|\nabla \omega(t)\|_{L^\infty}$.

Then, the Besov norm of the solution grows in an exponential way with respect to the norm of the transport field v. Nevertheless, if v is divergence-free then the $B^0_{\infty,r}$ norm of ω grows linearly in v: this was proved first by Vishik in [28], and then by Hmidi and Keraani in [21]. Here we generalize their result to the case when v is not divergence free. Of course, we will get a growth also on $\operatorname{div} v$, which is still suitable for our scopes (see Subsection 5.2).

Proposition 4.4. *Let us consider the linear transport equation* (50).

For any $\beta > 0$*, there exists a constant* C*, depending only on* d *and* β*, such that the following a priori estimate holds true:*

$$\|\omega(t)\|_{B^0_{\infty,1}} \leq C\Big(\|\omega_0\|_{B^0_{\infty,1}} + \|g\|_{L^1_t(B^0_{\infty,1})}\Big)\Big(1 + \mathcal{V}(t)\Big),$$

with $\mathcal{V}(t)$ *defined by*

$$\mathcal{V}(t) := \int_0^t \Big(\|\nabla v\|_{L^\infty} + \|\operatorname{div} v\|_{B^\beta_{\infty,\infty}}\Big)\, dt'.$$

Proof. We will follow the proof of [21]. Firstly we can write the solution ω of the transport equation (50) as a sum: $\omega = \sum_{k \geq -1} \omega_k$, with ω_k satisfying

$$\begin{cases} \partial_t \omega_k + v \cdot \nabla \omega_k = \Delta_k g, \\ \omega_k|_{t=0} = \Delta_k \omega_0. \end{cases} \tag{52}$$

We obviously have from above that

$$\|\omega_k(t)\|_{L^\infty} \leq \|\Delta_k \omega_0\|_{L^\infty} + \int_0^t \|\Delta_k g\|_{L^\infty}\, \mathrm{d}t'. \tag{53}$$

By classical transport estimates in Proposition 4.3, for any $\epsilon \in (0, 1)$, we have

$$\|\omega_k(t)\|_{B^\epsilon_{\infty,1}} \leq \Big(\|\Delta_k \omega_0\|_{B^\epsilon_{\infty,1}} + \|\Delta_k g\|_{L^1_t(B^\epsilon_{\infty,1})}\Big) \exp\Big(C\|\nabla v\|_{L^1_t(L^\infty)}\Big). \tag{54}$$

In order to get a priori estimates in Besov space $B^{-\epsilon}_{\infty,1}$, after applying the operator Δ_j to Equation (52) to get

$$\begin{cases} \partial_t (\Delta_j \omega_k) + v \cdot \nabla(\Delta_j \omega_k) = \Delta_j \Delta_k g + [v, \Delta_j] \cdot \nabla \omega_k, \\ (\Delta_j \omega_k)|_{t=0} = \Delta_j \Delta_k \omega_0. \end{cases} \tag{55}$$

We write the commutator $[v, \Delta_j] \cdot \nabla \omega_k$ as follows (recalling Bony's decomposition (20) and denoting $\widetilde{v} := v - \Delta_{-1} v$):

$$[v, \Delta_j] \cdot \nabla \omega_k = [T_{\widetilde{v}}, \Delta_j] \cdot \nabla \omega_k + T_{\Delta_j \nabla \omega_k} \widetilde{v} + R(\Delta_j \nabla \omega_k, \widetilde{v}) - \Delta_j (T_{\nabla \omega_k} \widetilde{v}) \\ - \Delta_j \mathrm{div}\,(R(\omega_k, \widetilde{v})) + \Delta_j R(\omega_k, \mathrm{div}\,\widetilde{v}) + [\Delta_{-1} v, \Delta_j] \cdot \nabla \omega_k .$$

Then, for all $\beta > \epsilon$, by use of Proposition 3.4 and Proposition 3.5, the L^∞-norm of all the above terms can be bounded by (for some nonnegative sequence $\left\|(c_j)_j\right\|_{\ell^1} = 1$):

$$C(d,\beta)\, 2^{j\epsilon}\, c_j\, \mathcal{V}'(t) \|\omega_k\|_{B^{-\epsilon}_{\infty,1}}, \quad \text{with } \mathcal{V}'(t) = \|\nabla v\|_{L^\infty} + \|\mathrm{div}\, v\|_{B^{\beta}_{\infty,\infty}} .$$

Thus, we apply the a priori estimate (53) to the equation (55), multiply both sides by $2^{-j\epsilon}$ and take ℓ^r-norm on j, and finally by Gronwall's lemma we have the following a priori estimate in the space $B^{-\epsilon}_{\infty,1}$:

$$\|\omega_k(t)\|_{B^{-\epsilon}_{\infty,1}} \leq \Big(\|\Delta_k \omega_0\|_{B^{-\epsilon}_{\infty,1}} + \|\Delta_k g\|_{L^1_t(B^{-\epsilon}_{\infty,1})} \Big) \exp\Big(C\mathcal{V}(t) \Big). \tag{56}$$

On the other side, one has the following, for some positive integer N to be determined hereafter:

$$\|\omega\|_{B^0_{\infty,1}} \leq \sum_{j,k \geq -1} \|\Delta_j \omega_k\|_{L^\infty} = \sum_{|j-k|<N} \|\Delta_j \omega_k\|_{L^\infty} + \sum_{|j-k| \geq N} \|\Delta_j \omega_k\|_{L^\infty} .$$

Estimate (53) implies

$$\sum_{|j-k|<N} \|\Delta_j \omega_k\|_{L^\infty} \leq N \sum_k \Big(\|\Delta_k \omega_0\|_{L^\infty} + \|\Delta_k g\|_{L^1_t(L^\infty)} \Big) \leq N \Big(\|\omega_0\|_{B^0_{\infty,1}} + \|g\|_{L^1_t(B^0_{\infty,1})} \Big),$$

while Estimates (54) and (56) entail the following (for some nonnegative sequence $(c_j) \in \ell^1$):

$$\|\Delta_j \omega_k\|_{L^\infty} \leq 2^{-\epsilon|k-j|} c_j \Big(\|\Delta_k \omega_0\|_{L^\infty} + \|\Delta_k g\|_{L^1_t(L^\infty)} \Big) \exp\Big(C\mathcal{V}(t) \Big),$$

which issues immediately

$$\sum_{|j-k| \geq N} \|\Delta_j \omega_k\|_{L^\infty} \leq 2^{-N\epsilon} \Big(\|\omega_0\|_{B^0_{\infty,1}} + \|g\|_{L^1_t(B^0_{\infty,1})} \Big) \exp\Big(C\mathcal{V}(t) \Big).$$

Therefore, for any $\beta > 0$, we can choose $\epsilon \in (0,1)$ and $N \in \mathbb{N}$ such that $\epsilon < \beta$ and $N\epsilon \log 2 \sim 1 + C\mathcal{V}(t)$. Thus the lemma follows from the above estimates. $\square$

5. Proof of the main results

We are now ready to tackle the proof of our main results, for which this section is devoted to. We will first focus on the proof of Theorem 2.1 and in the second part, instead, we will deal with Theorem 2.4.

5.1. Proof of the local in time well-posedness result

In this subsection we will prove Theorem 2.1. We will follow the standard procedure: in Step 1 we construct a sequence of approximate solutions satisfying uniform bounds, and in Step 2 we prove the convergence of this sequence.

For the sake of conciseness, we will present the proof just for $r = 1$, for which we can use classical time-dependent spaces $L^q_T(B^s_{\infty,1})$. The general case is just more technical, but it does not involve any novelty: it can be treated as in [18], by use of refined commutator and product estimates in Chemin–Lerner spaces.

The interpolation inequalities (17) and (18), the product estimates in Proposition 3.5 and the estimates for the composition of functions in Proposition 3.8 will be used thoroughly in the proof. In particular, noticing $\rho \in [\rho_*, \rho^*]$, we have the following product estimates for $s > 0$

$$\left\| \Big(a(\rho) - 1, b(\rho) - 1, \kappa(\rho) - \kappa(1) \Big) \right\|_{B^s_{\infty,1}} \le C \|\varrho\|_{B^s_{\infty,1}}, \quad \|uv\|_{B^s_{\infty,1}} \le C \|u\|_{B^s_{\infty,1}} \|v\|_{B^s_{\infty,1}}, \tag{57}$$

and for $s \ge 1$ and for two smooth functions f, g defined on $\mathbb{R}^+$, one has

$$\begin{aligned} \|\nabla^2 f(\rho) \otimes (\nabla g(\rho), u)\|_{B^s_{\infty,1}} &\le C \|\nabla^2 f(\rho)\|_{L^\infty} \|(\nabla g(\rho), u)\|_{B^s_{\infty,1}} \\ &\quad + C \|\nabla^2 f(\rho)\|_{B^s_{\infty,1}} \|(\nabla g(\rho), u)\|_{L^\infty} \\ &\le C \|\varrho\|_{B^{s+1}_{\infty,1}} \|(\nabla \varrho, u)\|_{B^s_{\infty,1}} + C \|(\varrho, u)\|_{B^s_{\infty,1}} \|\varrho\|_{B^{s+2}_{\infty,1}} \\ &\le C \|(\varrho, u)\|_{B^s_{\infty,1}} \|\varrho\|_{B^{s+2}_{\infty,1}}, \end{aligned} \tag{58}$$

$$\|\nabla^2 f(\rho) \otimes (\nabla g(\rho), u)\|_{L^2} \le \|\nabla^2 f(\rho)\|_{L^\infty} \|(\nabla g(\rho), u)\|_{L^2} \le C \|\varrho\|_{B^{s+1}_{\infty,1}} \|(\nabla \rho, u)\|_{L^2}, \tag{59}$$

$$\|\mathrm{div}\, (v \cdot \nabla u)\|_{B^{s-1}_{\infty,1}} \le C \|\nabla v\|_{B^{s-1}_{\infty,1}} \|\nabla u\|_{B^{s-1}_{\infty,1}} \text{ if } \mathrm{div}\, v = 0. \tag{60}$$

Let us make some simplifications in the coming proof. We always suppose the existence time $T^* \le 1$ and that all the constants appearing in the sequel, such as C, C_M, C_E, are bigger than 1. We always denote $f^n = f(\rho^n)$ and $\delta f^n = f(\rho^n) - f(\rho^{n-1})$ for functions $f = f(\rho)$.

5.1.1. Step 1: construction of a sequence of approximate solutions

As usual, after fixing $(\varrho^0, u^0, \nabla \pi^0) = (\varrho_0, u_0, 0)$, we consider inductively the n-th approximate solution (ϱ^n, u^n, π^n) to be the unique global solution of the following linear system:

$$\begin{cases} \partial_t \varrho^n + u^{n-1} \cdot \nabla \varrho^n - \mathrm{div}\, (\kappa^{n-1} \nabla \varrho^n) = 0, \\ \partial_t u^n + (u^{n-1} - \kappa^{n-1} (\rho^n)^{-1} \nabla \rho^n) \cdot \nabla u^n + \lambda^n \nabla \pi^n = h^{n-1}, \\ \mathrm{div}\, u^n = 0, \\ (\varrho^n, u^n)|_{t=0} = (\varrho_0, u_0), \end{cases} \tag{61}$$

where $b^{n-1} = b(\rho^{n-1})$, $\kappa^{n-1} = \kappa(\rho^{n-1})$, $\lambda^{n-1} = \lambda(\rho^{n-1})$ and

$$\begin{aligned} h^{n-1} = (\rho^{n-1})^{-1} \Big(\Delta b^{n-1} \nabla a^{n-1} + u^{n-1} \cdot \nabla^2 a^{n-1} + \nabla b^{n-1} \cdot \nabla^2 a^{n-1} \Big), \\ a^{n-1} = a(\rho^{n-1}). \end{aligned} \tag{62}$$

It is easy to see that by testing (61)$_2$ by $\rho^n u^n$, one should have the energy identity for u^n

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} \rho^n |u^n|^2 = \int_{\mathbb{R}^d} \rho^n h^{n-1} \cdot u^n. \tag{63}$$

In this paragraph, one denotes

$$M := \|(\varrho_0, u_0)\|_{B^s_{\infty,1}}, \qquad E_0 := \|\varrho_0\|_{L^2} + \|u_0\|_{L^2}.$$

We aim at proving that there exist a sufficiently small parameter τ (to be determined later), a positive time T^* (which may depend on τ), a positive constant C_M (which may depend on M) and a positive constant C_E such that the uniform estimates for the solution sequence $(\rho^n, u^n, \nabla\pi^n)$ hold:

$$\rho_* \le \rho^n := 1 + \varrho^n \,, \, \|\varrho^n\|_{L^\infty_{T^*}(B^s_{\infty,1})} \le C_M \,, \quad \|\varrho^n\|_{L^2_{T^*}(B^{s+1}_{\infty,1}) \cap L^1_{T^*}(B^{s+2}_{\infty,1})} \le \tau \,, \tag{64}$$

$$\|u^n\|_{L^\infty_{T^*}(B^s_{\infty,1})} \le C_M, \; \|u^n\|_{L^2_{T^*}(B^s_{\infty,1}) \cap L^1_{T^*}(B^s_{\infty,1})} \le \tau, \; \|\nabla\pi^n\|_{L^1_{T^*}(B^s_{\infty,1} \cap L^2)} \le \tau^{\alpha/2}, \tag{65}$$

$$\|\varrho^n\|_{L^\infty_{T^*}(L^2)} + \|\nabla\varrho^n\|_{L^2_{T^*}(L^2)} + \|u^n\|_{L^\infty_{T^*}(L^2)} \le C_E E_0, \tag{66}$$

with α defined later.

Firstly, by choosing small T^*, Estimates (64), (65) and (66) all hold true for $n = 0$. Next we suppose the $(n-1)$-th element $(\varrho^{n-1}, u^{n-1}, \nabla\pi^{n-1})$ to belong to Space E, defined as the set of the triplet $(\varrho, u, \nabla\pi)$ belonging to

$$\Big(C(\mathbb{R}^+; B^s_{\infty,1} \cap L^2) \cap L^2_{\mathrm{loc}}(H^1) \cap L^1_{\mathrm{loc}}(B^{s+2}_{\infty,1})\Big) \\ \times \Big(C(\mathbb{R}^+; B^s_{\infty,1} \cap L^2)\Big)^d \times \Big(L^1_{\mathrm{loc}}(B^s_{\infty,1}) \cap L^1_{\mathrm{loc}}(L^2)\Big)^d, \tag{67}$$

such that the inductive assumptions (64), (65) and (66) are satisfied. We then just have to show that the same holds true for the n-th unknown $(\varrho^n, u^n, \nabla\pi^n)$ defined by System (61).

According to Proposition 4.1 and Remark 4.2, ϱ^n belongs to $C(\mathbb{R}^+; B^s_{\infty,1}) \cap L^1_{\mathrm{loc}}(B^{s+2}_{\infty,1})$. On the other hand, since $u^{n-1} \in L^\infty_{\mathrm{loc}}(L^\infty)$, energy inequality (11) for ϱ^n follows and $\varrho^n \in C(\mathbb{R}^+; L^2) \cap L^2_{\mathrm{loc}}(H^1)$, such that

$$\|\varrho^n\|_{L^\infty_t(L^2)} + \|\nabla\varrho^n\|_{L^2_t(L^2)} \le C\|\varrho_0\|_{L^2}. \tag{68}$$

As in [18], we introduce ϱ_L to be the solution of the free heat equation with initial datum ϱ_0, which satisfies

$$\|\varrho_L\|_{L^\infty_T(B^s_{\infty,1})} + \|\varrho_L\|_{L^1_T(B^{s+2}_{\infty,1})} \le C_T \|\varrho_0\|_{B^s_{\infty,1}}, \quad \forall T > 0, \quad C_T \text{ depending on } T, \tag{69}$$

$$\|\varrho_L\|_{L^2_{T^*}(B^{s+1}_{\infty,1}) \cap L^1_{T^*}(B^{s+2}_{\infty,1})} \le \tau^2 \quad \text{for small enough } T^*. \tag{70}$$

Correspondingly, the remainder $\bar\varrho^n := \varrho^n - \varrho_L$ verifies the following system:

$$\begin{cases} \partial_t \bar{\varrho}^n + u^{n-1} \cdot \nabla \bar{\varrho}^n - \operatorname{div}(\kappa^{n-1} \nabla \bar{\varrho}^n) = -u^{n-1} \cdot \nabla \varrho_L + \operatorname{div}((\kappa^{n-1} - 1) \nabla \varrho_L), \\ \bar{\varrho}^n|_{t=0} = 0. \end{cases} \tag{71}$$

Proposition 4.1 and Remark 4.2 thus imply that

$$\|\bar{\varrho}^n\|_{L_t^\infty(B^s_{\infty,1}) \cap L_t^1(B^{s+2}_{\infty,1})} \leq \left(C^{n-1}(t)\, e^{C^{n-1}(t) K^{n-1}(t)} \right) \|f^n\|_{L_t^1(B^s_{\infty,1})},$$

where $C^{n-1}(t)$ depends on $\|\varrho^{n-1}\|_{L_t^\infty(B^s_{\infty,1})}$, and

$$K^{n-1}(t) := t + \|\nabla \kappa^{n-1}\|^2_{L_t^2(B^s_{\infty,1})},$$

$$f^n := -u^{n-1} \cdot \nabla \bar{\varrho}^n - u^{n-1} \cdot \nabla \varrho_L + \operatorname{div}((\kappa^{n-1} - 1) \nabla \varrho_L).$$

Inductive assumptions, the product estimates (57) and the estimate (70) entail hence

$$C^{n-1}(t)\, e^{C^{n-1}(t) K^{n-1}(t)} \leq C_K, \quad \forall t \in [0, T^*],\ C_K \text{ depending only on } M,$$

$$\begin{aligned} \|f^n\|_{L^1_{T^*}(B^s_{\infty,1})} &\leq C \|u^{n-1}\|_{L^2_{T^*}(B^s_{\infty,1})} \|\nabla \bar{\varrho}^n\|_{L^2_{T^*}(B^s_{\infty,1})} \\ &\quad + C C_M \tau^2 \leq C\, \tau\, \|\nabla \bar{\varrho}^n\|_{L^2_{T^*}(B^s_{\infty,1})} + C C_M \tau^2. \end{aligned}$$

Therefore by the interpolation inequality (17) with $s' = s + 1$, the following smallness statement pertaining to $\bar{\varrho}^n$

$$\|\bar{\varrho}^n\|_{L^2_{T^*}(B^{s+1}_{\infty,1})} \leq \|\bar{\varrho}^n\|_{L^\infty_{T^*}(B^s_{\infty,1})} + \|\bar{\varrho}^n\|_{L^1_{T^*}(B^{s+2}_{\infty,1})} \leq \tau^{3/2} \tag{72}$$

is verified, provided we choose small τ such that $C C_K C_M \tau^{\frac{1}{2}} \leq 1/2$. Hence inductive assumption (64) holds for ϱ^n, by view of the estimates (69) and (70) for ϱ_L.

We will bound u^n and $\nabla \pi^n$ in the following steps.

- Energy equality (63) holds and hence there holds

$$\|u^n\|^2_{L_t^\infty(L^2)} \leq C \|u_0\|^2_{L^2} + C \int_0^t \|h^{n-1}\|_{L^2} \|u^n\|_{L^2} \mathrm{d}t'.$$

We use (59) to bound $\|h^{n-1}\|_{L^2}$, and hence we have

$$\int_0^t \|h^{n-1}\|_{L^2} \|u^n\|_{L^2} \mathrm{d}t' \leq C \int_0^t \|\varrho^{n-1}\|_{B^{s+1}_{\infty,1}} \|(\nabla \rho^{n-1}, u^{n-1})\|_{L^2} \|u^n\|_{L^2} \mathrm{d}t'.$$

Therefore by Young's inequality one derives

$$\|u^n\|^2_{L_t^\infty(L^2)} \leq C \|u_0\|^2_{L^2} + C \int_0^t \|\varrho^{n-1}\|^2_{B^{s+1}_{\infty,1}} \|u^n\|^2_{L^2} \mathrm{d}t' + \int_0^t \|(\nabla \rho^{n-1}, u^{n-1})\|^2_{L^2} \mathrm{d}t'.$$

Thus Gronwall's inequality and the inductive assumptions (64) and (68) imply the estimate (66) for (ϱ^n, u^n), by choosing T^* sufficiently small.

- We apply the standard estimates for transport equation in Proposition 4.3 to the equation (61) for u^n and then the inductive assumptions and the product estimates (57) and (58) (on $\|h^{n-1}\|_{L^1_t(B^s_{\infty,1})}$) ensure that

$$\|u^n\|_{L^\infty_t(B^s_{\infty,1})} \le CC_M e^{CC_M\tau}\Big(\|u_0\|_{B^s_{p,r}} + \tau + \|\nabla\pi^n\|_{L^1_t(B^s_{\infty,1})}\Big) \le CC_M(1+\Pi^n), \quad (73)$$

where we have defined

$$\Pi^n := \|\nabla\pi^n\|_{L^1_{T^*}(B^s_{\infty,1})}.$$

- Consider the elliptic equation satisfied by π^n:

$$\operatorname{div}(\lambda^n\nabla\pi^n) = \operatorname{div}\Big(h^{n-1} - (u^{n-1} - \kappa^{n-1}(\rho^n)^{-1}\nabla\rho^n)\cdot\nabla u^n\Big).$$

Inductive assumptions, energy estimates (66) and product estimates (59) on h^{n-1} imply

$$\begin{aligned}
\|\nabla\pi^n\|_{L^1_{T^*}(L^2)} &\le C\Big\|h^{n-1} - (u^{n-1} - \kappa^{n-1}(\rho^n)^{-1}\nabla\rho^n)\cdot\nabla u^n\Big\|_{L^1_{T^*}(L^2)} \\
&\le C\|\varrho^{n-1}\|_{L^2_{T^*}(B^{s+1}_{\infty,1})}\|(\nabla\rho^{n-1}, u^{n-1})\|_{L^2_{T^*}(L^2)} \\
&\quad + \|u^n\|_{L^2_{T^*}(B^s_{\infty,1})}\|(u^{n-1}, \nabla\rho^n)\|_{L^2_{T^*}(L^2)} \\
&\le CC_E E_0(\tau + \tau\|u^n\|_{L^\infty_{T^*}(B^s_{\infty,1})}) \\
&\le CC_E C_M E_0\tau(1+\Pi^n), \quad \text{if } (T^*)^{1/2} \le \tau. \qquad (74)
\end{aligned}$$

- Now, recalling the interpolation inequality (17) with $s' = s - \frac{1}{2}$ entails (with some appropriated C_Π and $0 < \alpha < 1$)

$$\|\nabla\pi^n\|_{L^1_{T^*}(B^{s-1/2}_{\infty,1})} \le C\|\nabla\pi^n\|^\alpha_{L^1_{T^*}(L^2)}\|\nabla\pi^n\|^{1-\alpha}_{L^1_{T^*}(B^s_{\infty,1})} \le C_\Pi(1+\Pi^n)\tau^\alpha.$$

- Let's consider the following equation

$$\Delta\pi^n = \nabla\log\rho^n\cdot\nabla\pi^n + \rho^n\operatorname{div}\Big(h^{n-1} - (u^{n-1} - \kappa^{n-1}(\rho^n)^{-1}\nabla\rho^n)\cdot\nabla u^n\Big).$$

Notice by Proposition 3.5 one has (noticing $s - \frac{1}{2} > 0$)

$$\|\nabla\log\rho^n\cdot\nabla\pi^n\|_{L^1_{T^*}(B^{s-1}_{\infty,1})} \lesssim \|\nabla\log\rho^n\|_{L^\infty_{T^*}(B^{s-1}_{\infty,1})}\|\nabla\pi^n\|_{L^1_{T^*}(B^{s-\frac{1}{2}}_{\infty,1})}.$$

Thanks to product estimates (57), (58) and (60), for some (new) C_Π,

$$\|\Delta\pi^n\|_{L^1_{T^*}(B^{s-1}_{\infty,1})} \le CC_M\|\nabla\pi^n\|_{L^1_{T^*}(B^{s-1/2}_{\infty,1})} + CC_M\tau(1+\Pi^n) \le C_\Pi(1+\Pi^n)\tau^\alpha. \quad (75)$$

- By decomposing $\nabla\pi^n$ into low frequency part and high frequency part (and using Bernstein's inequality Lemma 3.2), one has

$$\Pi^n \lesssim \|\nabla\Delta_{-1}\pi^n\|_{L^1_{T^*}(L^2)} + \|\Delta\pi^n\|_{L^1_{T^*}(B^{s-1}_{\infty,1})} \lesssim \|\nabla\pi^n\|_{L^1_{T^*}(L^2)} + \|\Delta\pi^n\|_{L^1_{T^*}(B^{s-1}_{\infty,1})}.$$

Thus, the above two estimates (74) and (75) imply, for τ and T^* small enough, the inductive assumption (65) for π^n. Furthermore, (73) entails the inductive assumption (65) for u^n.

5.1.2. Step 2: convergence of the approximate solutions' sequence

We turn now to establish that the above sequence converges to a "true" solution of system (6). Let us introduce the difference sequence

$$(\delta\varrho^n, \delta u^n, \nabla\delta\pi^n) = (\varrho^n - \varrho^{n-1}, u^n - u^{n-1}, \nabla\pi^n - \nabla\pi^{n-1}), \quad n \geq 1.$$

When $n \geq 2$, it verifies the following system:

$$\begin{cases} \partial_t\delta\varrho^n + u^{n-1}\cdot\nabla\delta\varrho^n - \mathrm{div}\,(\kappa^{n-1}\nabla\delta\varrho^n) = F^{n-1}, \\ \partial_t\delta u^n + (u^{n-1} - \kappa^{n-1}\nabla\log\rho^n)\cdot\nabla\delta u^n + \lambda^n\nabla\delta\pi^n = H_e^{n-1}, \\ \mathrm{div}\,\delta u^n = 0, \\ (\delta\varrho^n, \delta u^n)|_{t=0} = (0,0), \end{cases} \tag{76}$$

where we have set

$$\begin{aligned} F^{n-1} &= -\delta u^{n-1}\cdot\nabla\varrho^{n-1} + \mathrm{div}\,(\delta\kappa^{n-1}\nabla\varrho^{n-1}), \\ H_e^{n-1} &= \delta h^{n-1} - (\delta u^{n-1} - \delta\kappa^{n-1}\nabla\log\rho^n - \kappa^{n-2}\nabla\delta(\log\rho)^n)\cdot\nabla u^{n-1} - \delta\lambda^n\nabla\pi^{n-1}, \end{aligned}$$

with $\delta h^{n-1} = h^{n-1} - h^{n-2}$.

We will consider the difference sequence in the energy space. Let's do some analysis first: one needs H_e^{n-1} in $L^1_{T^*}(L^2)$ and hence

$$(\rho^{n-1})^{-1}\Delta\delta b^{n-1}\nabla a^{n-1} \quad \text{and} \quad (\rho^{n-1})^{-1}\nabla b^{n-2}\cdot\nabla^2\delta a^{n-1} \text{ in } L^1_{T^*}(L^2).$$

We only have $\nabla\varrho^n$ in $L^\infty_{T^*}(L^\infty)$, and thus we need $\nabla^2\delta\varrho^n$ in $L^1_{T^*}(L^2)$: this property follows from the energy inequality of the equation of $\nabla\delta\varrho^n$

$$\begin{aligned} &\partial_t\nabla\delta\varrho^n + u^{n-1}\cdot\nabla^2\delta\varrho^n - \mathrm{div}\,(\kappa^{n-1}\nabla^2\delta\varrho^n) \\ &\quad = -\nabla\delta\varrho^n\cdot\nabla u^{n-1} + \mathrm{div}\,(\nabla\delta\varrho^n\otimes\nabla\kappa^{n-1}) + \nabla F^{n-1}. \end{aligned} \tag{77}$$

In the above, the first two terms of the right-hand side are of lower order, while the third one is in $L^2_{\mathrm{loc}}(H^{-1})$, thus taking L^2 inner product works.

Now we begin to make the above analysis in detail.

Since $\delta\varrho^n \in E$ (recall (67) for its definition), the energy equality for Equation $(76)_1$ holds for $n \geq 2$:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}^d}|\delta\varrho^n|^2\mathrm{d}x+\int\limits_{\mathbb{R}^d}\kappa^{n-1}|\nabla\delta\varrho^n|^2\mathrm{d}x\\
&\qquad\qquad=-\int\limits_{\mathbb{R}^d}\delta u^{n-1}\cdot\nabla\rho^{n-1}\,\delta\varrho^n\mathrm{d}x-\int\limits_{\mathbb{R}^d}\delta\kappa^{n-1}\nabla\rho^{n-1}\cdot\nabla\delta\varrho^n\mathrm{d}x\\
&\qquad\qquad\le\tau^2\|\delta u^{n-1}\|^2_{L^2(\mathbb{R}^d)}+\tau^{-2}\|\nabla\rho^{n-1}\|^2_{L^\infty(\mathbb{R}^d)}\|\delta\varrho^n\|^2_{L^2(\mathbb{R}^d)}\\
&\qquad\qquad\quad+C_\varepsilon\|\delta\kappa^{n-1}\|^2_{L^2(\mathbb{R}^d)}\|\nabla\rho^{n-1}\|^2_{L^\infty(\mathbb{R}^d)}+\varepsilon\|\nabla\delta\varrho^n\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Thus integration in time and the uniform estimates for the solutions' sequence give (choosing ε small enough)

$$\begin{aligned}
&\|\delta\varrho^n\|_{L^\infty_{T^*}(L^2)}+\|\nabla\delta\varrho^n\|_{L^2_{T^*}(L^2)}\\
&\qquad\le C(\|\delta\varrho^{n-1}\|_{L^\infty_{T^*}(L^2)}+\|\delta u^{n-1}\|_{L^2_{T^*}(L^2)})\,\tau.
\end{aligned}\tag{78}$$

Similarly, energy equality holds for $\nabla\delta\varrho^n$, $n\ge 2$ (in fact, it's not clear that $\delta\varrho^1\in L^2_{\mathrm{loc}}(H^2)$):

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}^d}|\nabla\delta\varrho^n|^2\mathrm{d}x+\int\limits_{\mathbb{R}^d}\kappa^{n-1}|\nabla^2\delta\varrho^n|^2\mathrm{d}x\\
&=-\int\limits_{\mathbb{R}^d}\Big(\nabla\delta\varrho^n\cdot\nabla u^{n-1}\cdot\nabla\delta\varrho^n+\nabla\delta\varrho^n\cdot\nabla^2\delta\varrho^n\cdot\nabla\kappa^{n-1}+F^{n-1}\Delta\delta\varrho^n\Big)\mathrm{d}x.
\end{aligned}$$

Integrating in time and the inductive assumptions also imply

$$\begin{aligned}
&\|\nabla\delta\varrho^n\|_{L^\infty_{T^*}(L^2)}+\|\nabla^2\delta\varrho^n\|_{L^2_{T^*}(L^2)}\\
&\qquad\le C\tau\|(\delta\varrho^{n-1},\delta u^{n-1})\|_{L^\infty_{T^*}(L^2)}+CC_M\|\nabla\delta\varrho^{n-1}\|_{L^2_{T^*}(L^2)}.
\end{aligned}$$

By controlling $\|\nabla\delta\varrho^{n-1}\|_{L^2_{T^*}(L^2)}$ above by (78), one sums up these two inequalities, obtaining

$$\begin{aligned}
&\|\delta\varrho^n\|_{L^\infty_{T^*}(H^1)}+\|\nabla\delta\varrho^n\|_{L^2_{T^*}(H^1)}\\
&\qquad\le CC_M\tau\|(\delta\varrho^{n-1},\delta\varrho^{n-2},\delta u^{n-1},\delta u^{n-2})\|_{L^\infty_{T^*}(L^2)}.
\end{aligned}\tag{79}$$

Now we turn to δu^n. We rewrite δh^{n-1} as

$$\begin{aligned}
&\frac{1}{\rho^{n-1}}\Big(\Delta\delta b^{n-1}\nabla a^{n-1}+\Delta b^{n-2}\nabla\delta a^{n-1}+\delta u^{n-1}\cdot\nabla^2a^{n-1}\\
&\quad+u^{n-2}\cdot\nabla^2\delta a^{n-1}+\nabla\delta b^{n-1}\cdot\nabla^2a^{n-1}+\nabla b^{n-2}\cdot\nabla^2\delta a^{n-1}\Big)\\
&\quad+\Big((\rho^{n-1})^{-1}-(\rho^{n-2})^{-1}\Big)(\Delta b^{n-2}\cdot\nabla a^{n-2}+u^{n-2}\cdot\nabla^2a^{n-2}+\nabla b^{n-2}\cdot\nabla^2a^{n-2}).
\end{aligned}$$

From the inductive estimates we also have that

$$\begin{aligned}\|\delta h^{n-1}\|_{L^1_{T^*}(L^2)} \le C C_M \tau (\|\delta\varrho^{n-1}\|_{L^2_{T^*}(H^2)} + \|\delta u^{n-1}\|_{L^\infty_{T^*}(L^2)}) \\ + C C_E E_0 \tau \|\delta\varrho^{n-1}\|_{L^\infty_{T^*}(L^2)},\end{aligned}$$

and

$$\begin{aligned}\|H_e^{n-1}\|_{L^1_{T^*}(L^2)} \le C(C_M + C_E E_0)\tau (\|\delta\varrho^{n-1}\|_{L^2_{T^*}(H^2)} + \|\delta u^{n-1}\|_{L^\infty_{T^*}(L^2)}) \\ + C\tau^{\alpha/2}\|\delta\varrho^n\|_{L^\infty_{T^*}(L^2)}.\end{aligned}$$

By view of the density equation for ρ^n and $\operatorname{div}\delta u^n = 0$, one has

$$\|\delta u^n\|_{L^\infty_{T^*}(L^2)} \le C\|H_e^{n-1}\|_{L^1_{T^*}(L^2)}. \tag{80}$$

Combining Estimate (79) and (80) entails, for sufficiently small τ (depending only on d, C_M, C_E, E_0),

$$\begin{aligned}&\|\delta\varrho^n\|_{L^\infty_{T^*}(H^1)\cap L^2_{T^*}(H^2)} + \|\delta u^n\|_{L^\infty_{T^*}(L^2)} \\ &\le \frac{1}{6}\|(\delta\varrho^{n-1}, \delta\varrho^{n-2}, \delta\varrho^{n-3}, \delta u^{n-1}, \delta u^{n-2}, \delta u^{n-3})\|_{L^\infty_{T^*}(L^2)}.\end{aligned}$$

Thus $\sum\|(\delta\varrho^n, \delta u^n)\|_{L^\infty_{T^*}(L^2)}$ converges. Since $\delta\varrho^n \in C(\mathbb{R}^+; L^2)$, the Cauchy sequences $\{\varrho^n\}$ and $\{u^n\}$ converge respectively to ϱ and u in $C([0,T^*]; L^2)$. It is also easy to see that

$$\sum_{n\ge 2}\|\delta\varrho^n\|_{L^\infty_{T^*}(H^1)\cap L^2_{T^*}(H^2)}, \quad \sum_{n\ge 2}\|\delta h^n\|_{L^1_{T^*}(L^2)}, \quad \sum_{n\ge 2}\|H_e^{n-1}\|_{L^1_{T^*}(L^2)} < +\infty.$$

Rewrite the elliptic equation for π^n

$$\begin{aligned}\operatorname{div}(\lambda^n\nabla\delta\pi^n) &= \operatorname{div} H_e^{n-1} - \operatorname{div}((u^{n-1} - \kappa^{n-1}\nabla\log\rho^n)\cdot\nabla\delta u^n), \\ &= \operatorname{div} H_e^{n-1} - \operatorname{div}\Big(\delta u^n\cdot\nabla(u^{n-1} - \kappa^{n-1}\nabla\log\rho^n) + \delta u^n \operatorname{div}(\kappa^{n-1}\nabla\log\rho^n)\Big).\end{aligned}$$

We hence get

$$\|\nabla\delta\pi^n\|_{L^1_{T^*}(L^2)} \le C(\|H_e^{n-1}\|_{L^1_{T^*}(L^2)} + C_M\|\delta u^n\|_{L^\infty_{T^*}(L^2)}).$$

Thus $\sum_2^\infty\|\nabla\delta\pi^n\|_{L^1_{T^*}(L^2)}$ also converges and hence $\nabla\pi^n$ converges to the unique limit $\nabla\pi$ in $L^1_{T^*}(L^2)$.

Finally, one easily checks that the limit $(\rho, u, \nabla\pi)$ solves System (6) and is in $E(T^*)$ by Fatou property. The proof of the uniqueness and the stability is quite similar and we omit it.

5.2. Lower bounds for the lifespan in dimension $d = 2$

In this section, we aim to get a lower bound for the lifespan of the solution in the case of dimension $d = 2$. The idea is to resort to the vorticity in order to control the high frequencies of the velocity field, as done in [15] in the context of incompressible Euler equations with variable density. The key to the proof will be bounding, by use of Proposition 4.4, the $B^0_{\infty,1}$ norm of the vorticity linearly (but not exponentially) with respect to the velocity field. Let us just mention that in the proof we always keep in mind that $\|\rho_0 - 1\|_{B^1_{\infty,1}} \le 1$ should be small and we will pay attention to the explicit dependence on it.

Let us define the (scalar) vorticity ω of the fluid as in the classical case:

$$\omega := \partial_1 u^2 - \partial_2 u^1 \equiv \partial_1 v^2 - \partial_2 v^1 . \tag{81}$$

According to $(1)_2$, it satisfies the following transport equation:

$$\partial_t \omega + v \cdot \nabla \omega + \omega \Delta b + \nabla \lambda \wedge \nabla \Pi = 0 , \tag{82}$$

where (recalling the change of variables (4) and (5))

$$v = u + \nabla b, \quad a = a(\rho),\ b = b(\rho),\ \lambda = \lambda(\rho),$$
$$\nabla \Pi = \nabla \pi + \nabla \partial_t a, \quad \nabla \lambda \wedge \nabla \Pi = \partial_1 \lambda\, \partial_2 \Pi - \partial_2 \lambda\, \partial_1 \Pi .$$

Similarly as in [18], let us introduce the following notations:

$$R_0 = \|\varrho_0\|_{B^1_{\infty,1}} \le 1, \quad U_0 = \|u_0\|_{B^1_{\infty,1}},$$
$$R(t) = \|\varrho\|_{L^\infty_t(B^1_{\infty,1})}, \quad S(t) = \|\varrho\|_{L^1_t(B^3_{\infty,1})}, \quad U(t) = \|u\|_{L^\infty_t(B^1_{\infty,1})}.$$

First of all, we apply Proposition 4.1 to the density equation $(6)_1$: it is easy to see that we get, for some $\epsilon \in (0, 1)$ (denoting $S'(t) = \|\varrho(t)\|_{B^3_{\infty,1}}$ and noticing the product estimate (57) and the interpolation inequality (17)),

$$R + S \le C(1 + R^{\frac{2}{\epsilon(1-\epsilon)}}) \left(R_0 + \int_0^t \left(U R^{1/2} (S')^{1/2} + \left(1 + R^{\frac{2-\epsilon}{1+\epsilon}} (S')^{\frac{\epsilon}{1+\epsilon}}\right) R + R^{3/2} (S')^{1/2} \right) \mathrm{d}t' \right). \tag{83}$$

Hence by use of Young's inequality and choosing $\epsilon = 1/2$, the above estimate (83) becomes

$$R + S \le C(1 + R^8) R_0 + C(1 + R^{16}) \int_0^t \left(U^2 R + R + R^3 \right) \mathrm{d}t' .$$

If we define now

$$T_R := \sup\left\{t > 0 \;\middle|\; R \le 2,\ \int_0^t R^3\,\mathrm{d}t' \le 2\,R_0\right\}, \tag{84}$$

then by Gronwall inequality, for all $t \in [0, T_R]$ we infer the estimate

$$R + S \le C\,R_0 \exp\left(C\int_0^t (1+U^2)\mathrm{d}t'\right). \tag{85}$$

We now estimate the velocity field. Similar as in (57)–(60), let us summarize the following inequalities for the non-linear terms in the momentum equation, which will be frequently used in the sequel:

$$\|\nabla^2 b(\rho)\|_{B^1_{\infty,1}} \lesssim \|b\|_{B^3_{\infty,1}} \lesssim \|\varrho\|_{B^3_{\infty,1}} = S'; \tag{86}$$

$$\begin{aligned} \|\Delta b \nabla a\|_{L^2} &\lesssim \|b\|_{B^2_{\infty,1}} \|\nabla a\|_{L^2} \lesssim \|\varrho\|_{B^2_{\infty,1}} \|\nabla\rho\|_{L^2} \\ &\lesssim R^{1/2}(S')^{1/2}\|\nabla\rho\|_{L^2} \le R\|\nabla\rho\|^2_{L^2} + S'; \end{aligned} \tag{87}$$

$$\|(u+\nabla b)\cdot\nabla u\|_{L^2} \lesssim \|\nabla u\|_{L^\infty}(\|\nabla\rho\|_{L^2} + \|u\|_{L^2}) \lesssim U(\|\nabla\rho\|_{L^2} + \|u\|_{L^2}). \tag{88}$$

Similarly as the above Inequality (87), one has also

$$\|\nabla b\cdot\nabla^2 a\|_{L^2} \lesssim R\|\nabla\rho\|^2_{L^2} + S', \quad \|u\cdot\nabla^2 a\|_{L^2} \lesssim R\|u\|^2_{L^2} + S'. \tag{89}$$

Now, by separating low and high frequencies, we find the following bound for the velocity:

$$U(t) \le C\left(\|u\|_{L^2} + \|\omega\|_{B^0_{\infty,1}}\right). \tag{90}$$

From the energy inequality for Equation (12) of u, i.e.

$$\|u(t)\|_{L^2} \le C\left(\|u_0\|_{L^2} + \int_0^t \|\mathrm{div}\,(v\otimes\nabla a)\|_{L^2}\,\mathrm{d}t'\right),$$

due to Inequalities (87) and (89), it follows that

$$\|u(t)\|_{L^2} \le C\left(\|u_0\|_{L^2} + \int_0^t \left(R(\|\nabla\rho\|^2_{L^2} + \|u\|^2_{L^2}) + S'\right)\mathrm{d}t'\right). \tag{91}$$

Now, applying Proposition 4.4 with $\beta = 1$ to Equation (82), we find

$$\|\omega(t)\|_{B^0_{\infty,1}} \lesssim \left(\|\omega_0\|_{B^0_{\infty,1}} + \int_0^t \left\|\nabla\lambda \wedge \nabla\Pi + \omega\,\Delta b\right\|_{B^0_{\infty,1}} \mathrm{d}t'\right) \times \left(1 + \int_0^t \left(\|\nabla u\|_{L^\infty} + \|\nabla^2 b\|_{B^1_{\infty,1}}\right) \mathrm{d}t'\right).$$

By use of Bony's paraproduct decomposition (see also [15]), one has

$$\|\nabla\lambda \wedge \nabla\Pi\|_{B^0_{\infty,1}} \lesssim \|\nabla\rho\|_{B^0_{\infty,1}} \|\nabla\Pi\|_{B^0_{\infty,1}} \qquad \text{and} \qquad \|\omega\,\Delta b\|_{B^0_{\infty,1}} \lesssim \|\omega\|_{B^0_{\infty,1}} \|\Delta b\|_{B^1_{\infty,1}};$$

hence, by virtue of the relation $\|\omega\|_{B^0_{\infty,1}} \lesssim U$, we get

$$\|\omega(t)\|_{B^0_{\infty,1}} \leq C\left(U_0 + \int_0^t \left(R\|\nabla\Pi\|_{B^0_{\infty,1}} + U\,S'\right) \mathrm{d}t'\right)\left(1 + \int_0^t \|\nabla u\|_{L^\infty} \mathrm{d}t' + S\right). \tag{92}$$

It remains us to deal with the pressure term. First of all, from the density equation we have

$$\|\nabla\Pi\|_{B^0_{\infty,1}} \leq \|\nabla\pi\|_{B^0_{\infty,1}} + \|\partial_t \nabla a\|_{B^0_{\infty,1}} \lesssim \|\nabla\pi\|_{B^0_{\infty,1}} + U\,S' + S'.$$

We next bound π, which satisfies the following elliptic equation:

$$\mathrm{div}\,(\lambda\nabla\pi) = \mathrm{div}\,(h - v \cdot \nabla u) = \mathrm{div}\,(h - u \cdot \nabla v + u\mathrm{div}\, v).$$

Similarly as the end of Step 1, Subsection 5.1, decomposing $\nabla\pi$ into the high and low frequency parts yields to

$$\begin{aligned}
\|\nabla\pi\|_{B^1_{\infty,1}} &\lesssim \|\nabla\Delta_{-1}\pi\|_{L^\infty} + \|\Delta\pi\|_{B^0_{\infty,1}} \\
&\lesssim \|\nabla\pi\|_{L^2} + \|\lambda^{-1}[-\nabla\lambda \cdot \nabla\pi + \mathrm{div}\,(h - v \cdot \nabla u)]\|_{B^0_{\infty,1}} \\
&\lesssim \|h - v \cdot \nabla u\|_{L^2} + \|\nabla\rho\|_{B^0_{\infty,1}} \|\nabla\pi\|_{B^{\frac{1}{2}}_{\infty,1}} \\
&\quad + (1 + \|\nabla\rho\|_{B^0_{\infty,1}})(\|h\|_{B^1_{\infty,1}} + \|\mathrm{div}\,(v \cdot \nabla u)\|_{B^0_{\infty,1}}).
\end{aligned}$$

By the interpolation inequality (18) with $s'' = \frac{1}{2}$ and Young's inequality, one derives that, for some $\delta > 1$,

$$\|\nabla\pi\|_{B^1_{\infty,1}} \leq C\left((1 + R^\delta)\|h - (u + \nabla b) \cdot \nabla u\|_{L^2} + (1 + R)\left(\|h\|_{B^1_{\infty,1}} + \|\mathrm{div}\,(v \cdot \nabla u)\|_{B^0_{\infty,1}}\right)\right).$$

Then, by the product estimates in (86)–(89), one finally bounds $\nabla\pi$ as follows:

$$\|\nabla\pi\|_{B^1_{\infty,1}} \le C(1+R^\delta)\Big(R(\|\nabla\rho\|^2_{L^2}+\|u\|^2_{L^2})+U(\|\nabla\rho\|_{L^2}+\|u\|_{L^2}) \\ +(1+R^2)(US'+S'+U^2)\Big).$$

Let us define

$$X(t) := U(t)+\|u(t)\|_{L^2} = \|u(t)\|_{L^2\cap B^1_{\infty,1}}.$$

So we get

$$\|\nabla\Pi\|_{B^0_{\infty,1}},\ \|\nabla\pi\|_{B^1_{\infty,1}} \le C\left(1+R^{\delta+2}\right)\left(\|\nabla\rho\|^2_{L^2}+S'+X^2+XS'\right).$$

Therefore, Estimate (92) for the vorticity becomes (noticing there is a "coefficient" R before $\|\nabla\Pi\|_{B^0_{\infty,1}}$)

$$\|\omega(t)\|_{B^0_{\infty,1}} \lesssim \left(X_0+\int_0^t(1+R^{\delta+3})\left(R\|\nabla\rho\|^2_{L^2}+RS'+RX^2+XS'\right)\mathrm{d}t'\right) \\ \times\left(1+S+\int_0^t X\mathrm{d}t'\right),$$

with $X(0)=X_0$. Keeping in mind (85), from relation (90) we finally find, for all $t\in[0,T_R]$,

$$X(t)\le C\left(X_0+R_0\mathrm{e}^{C\int_0^t(1+X^2)}\left(\int_0^t\|\nabla\rho\|^2_{L^2}\mathrm{d}t'+1\right)+\mathrm{e}^{C\int_0^t(1+X^2)}\int_0^t XS'\,\mathrm{d}t'\right) \\ \times\left(1+S+\int_0^t X\mathrm{d}t'\right).$$

Let us define now T_X as the quantity

$$T_X := \sup\left\{t>0\ \middle|\ R_0\mathrm{e}^{C\int_0^t(1+X^2)}\le 2,\quad \mathrm{e}^{C\int_0^t(1+X^2)}\int_0^t XS'\,\mathrm{d}t'\le 2X_0\right\};\tag{93}$$

then notice that, in $[0,T_X]$, one has in particular (noticing that $S\le CR_0\mathrm{e}^{C\int_0^t(1+X^2)}\le 2C$ by (85) and $\|\nabla\rho\|_{L^2(L^2)}\le C\|\varrho_0\|_{L^2}$)

$$X(t)\le C(1+\|\varrho_0\|^2_{L^2}+X_0)\left(1+\int_0^t X\mathrm{d}t'\right),\ \forall t\in[0,T_R]\cap[0,T_X].$$

Therefore, defining $\Gamma_0 := (1 + \|\varrho_0\|_{L^2}^2 + X_0)$, then by Gronwall's lemma we get

$$X(t) \le C\Gamma_0 \, e^{C\Gamma_0 t} \qquad \text{and} \qquad R(t) + S(t) \le C \; R_0 \exp\left(C\Gamma_0^2 \, e^{C\,\Gamma_0\,t} \right) \text{ on } [0, T_R] \cap [0, T_X], \tag{94}$$

where we used also relation (85).

Now, by a standard bootstrap argument, we insert the previous estimates (94) on $(R, S, X)(t)$ into the conditions in (84) and (93) defining the times T_R and T_X respectively: after a quite straightforward calculation (which we omit here), one can check that, for $c > 0$ small enough, these conditions are fulfilled on $[0, T]$ with $T > 0$ given by relation (16).

This completes the proof of Theorem 2.4.

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References

[1] T. Alazard, Low Mach number limit of the full Navier–Stokes equations, Arch. Ration. Mech. Anal. 180 (1) (2006) 1–73.

[2] S.N. Antontsev, A.V. Kazhikhov, V.N. Monakhov, Boundary Value Problems in Mechanics of Nonhomogeneous Fluids, Studies in Mathematics and Its Applications, vol. 22, North-Holland Publishing Co., Amsterdam, 1990, translated from the Russian.

[3] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren der Mathematischen Wissenschaften, vol. 343, Springer-Verlag, 2011.

[4] H. Beirão da Veiga, R. Serapioni, A. Valli, On the motion of nonhomogeneous fluids in the presence of diffusion, J. Math. Anal. Appl. 85 (1) (1982) 179–191.

[5] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Ann. Sci. Éc. Norm. Supér. (4) 14 (2) (1981) 209–246.

[6] D. Bresch, B. Desjardins, Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model, Comm. Math. Phys. 238 (1–2) (2003) 211–223.
[7] D. Bresch, B. Desjardins, On the existence of global weak solutions to the Navier–Stokes equations for viscous compressible and heat conducting fluids, J. Math. Pures Appl. (9) 87 (1) (2007) 57–90.
[8] D. Bresch, B. Desjardins, C.-K. Lin, On some compressible fluid models: Korteweg, lubrication, and shallow water systems, Comm. Partial Differential Equations 28 (3–4) (2003) 843–868.
[9] D. Bresch, H. El Essoufi, M. Sy, Effect of density dependent viscosities on multiphasic incompressible fluid models, J. Math. Fluid Mech. 9 (3) (2007) 377–397.
[10] J.-Y. Chemin, N. Lerner, Flot de champs de vecteurs non lipschitziens et équations de Navier–Stokes, J. Differential Equations 121 (2) (1995) 314–328.
[11] H. Beirão da Veiga, Diffusion on viscous fluids. Existence and asymptotic properties of solutions, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 10 (2) (1983) 341–355.
[12] H. Beirão da Veiga, A. Valli, On the Euler equations for nonhomogeneous fluids. I, Rend. Semin. Mat. Univ. Padova 63 (1980) 151–168.
[13] H. Beirão da Veiga, A. Valli, On the Euler equations for nonhomogeneous fluids. II, J. Math. Anal. Appl. 73 (2) (1980) 338–350.
[14] R. Danchin, On the well-posedness of the incompressible density-dependent Euler equations in the L^p framework, J. Differential Equations 248 (8) (2010) 2130–2170.
[15] R. Danchin, F. Fanelli, The well-posedness issue for the density-dependent Euler equations in endpoint Besov spaces, J. Math. Pures Appl. 96 (3) (2011) 253–278.
[16] R. Danchin, X. Liao, On the well-posedness of the full low-Mach number limit system in general critical Besov spaces, Commun. Contemp. Math. 14 (3) (2012) 1250022.
[17] P. Embid, Well-posedness of the nonlinear equations for zero Mach number combustion, Comm. Partial Differential Equations 12 (11) (1987) 1227–1283.
[18] F. Fanelli, X. Liao, The well-posedness issue for an inviscid zero-Mach number system in general Besov spaces, Asymptot. Anal. 93 (1–2) (2015) 115–140.
[19] E. Feireisl, A. Novotný, Singular Limits in Thermodynamics of Viscous Fluids, Advances in Mathematical Fluid Mechanics, Birkhäuser Verlag, Basel, 2009.
[20] D.A. Frank-Kamenetskii, Diffusion and Heat Transfer in Chemical Kinetics, Plenum, New York/London, 1969.
[21] T. Hmidi, S. Keraani, Incompressible viscous flows in borderline Besov spaces, Arch. Ration. Mech. Anal. 189 (2) (2008) 283–300.
[22] A.V. Kazhikhov, Sh. Smagulov, The correctness of boundary value problems in a certain diffusion model of an inhomogeneous fluid, Dokl. Akad. Nauk SSSR 234 (2) (1977) 330–332.
[23] X. Liao, A global existence result for a zero Mach number system, J. Math. Fluid Mech. 16 (1) (2014) 77–103.
[24] P.-L. Lions, Mathematical Topics in Fluid Mechanics, vol. 1. Incompressible Models, Oxford Lecture Series in Mathematics and Its Applications, vol. 3, The Clarendon Press Oxford University Press, Oxford Science Publications, New York, 1996.
[25] A. Majda, Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables, Applied Mathematical Sciences, vol. 53, Springer-Verlag, New York, 1984.
[26] P. Secchi, On the motion of viscous fluids in the presence of diffusion, SIAM J. Math. Anal. 19 (1) (1988) 22–31.
[27] M. Sy, A remark on the Kazhikhov–Smagulov type model: the vanishing initial density, Appl. Math. Lett. 18 (12) (2005) 1351–1358.
[28] M. Vishik, Hydrodynamics in Besov spaces, Arch. Ration. Mech. Anal. 145 (3) (1998) 197–214.
[29] W. Wolibner, Un theorème sur l'existence du mouvement plan d'un fluide parfait, homogène, incompressible, pendant un temps infiniment long, Math. Z. 37 (1) (1933) 698–726.
[30] R.K. Zeytounian, Theory and Applications of Viscous Fluid Flows, Springer-Verlag, Berlin, 2004.