Bull. London Math. Soc. 44 (2012) 353–366
doi:10.1112/blms/bdr102

Riesz potentials and p-superharmonic functions in Lie groups of Heisenberg type

Nicola Garofalo and Jeremy T. Tyson

Abstract

We prove a superposition principle for Riesz potentials of nonnegative continuous functions on Lie groups of Heisenberg type. More precisely, we show that the Riesz potential

$$R_\alpha(\rho)(g) = \int_{\mathbb{G}} N(g^{-1}g')^{\alpha-Q}\rho(g')\,dg'$$

of a nonnegative function $\rho \in C_0(\mathbb{G})$ on a group $\mathbb{G}$ of Heisenberg type is necessarily either p-subharmonic or p-superharmonic, depending on p and α. Here N denotes the anisotropic homogeneous norm on such groups introduced by Kaplan. This result extends to a wide class of nonabelian stratified Lie groups a recent remarkable superposition result of Crandall and Zhang.

1. *Introduction*

The study of Riesz potentials occupies a central position in classical analysis and potential theory; see [**16**, **18**]. A basic result states that if one considers the Newtonian potential of $\rho \in C_0(\mathbb{R}^n)$, $n \geqslant 3$,

$$I_2(\rho)(x) = c_n \int_{\mathbb{R}^n} \frac{\rho(y)}{|x-y|^{n-2}}\,dy,$$

then $-\Delta(I_2(\rho)) = \rho$, in other words the operator I_2 is the inverse of minus the Laplacian. In particular, when $\rho \geqslant 0$, then $I_2(\rho)$ is a nonnegative superharmonic function in $\mathbb{R}^n$. Vice versa, the F. Riesz decomposition theorem states that every nonnegative superharmonic function in $\mathbb{R}^n$ arises, modulo harmonic functions, as the Newtonian potential of a measure; see [**13**]. More generally, the Riesz potential

$$I_\alpha(\rho)(x) = c_{n,\alpha} \int_{\mathbb{R}^n} \frac{\rho(y)}{|x-y|^{n-\alpha}}\,dy, \quad 0 < \alpha < n,$$

provides an inverse for the fractional power of the Laplacian. One has in fact the following well-known identities in $\mathcal{S}'(\mathbb{R}^n)$; see [**18**]:

$$(-\Delta)^{\alpha/2}(I_\alpha(\rho)) = I_\alpha((-\Delta)^{\alpha/2}\rho) = \rho.$$

What is remarkable is that the Riesz potentials, which as we have seen are intrinsically connected to a linear operator, the Laplacian, also interact with a highly nonlinear operator, namely, the p-Laplacian

$$\Delta_p f = \operatorname{div}(|\nabla f|^{p-2}\nabla f), \quad 1 < p < \infty.$$

This was discovered by Lindqvist and Manfredi [**17**], where the authors proved the following surprising result.

Received 16 February 2011; revised 7 July 2011; published online 4 December 2011.

2010 *Mathematics Subject Classification* 53C17 (primary), 31C15, 31C45, 31E05, 43A80 (secondary).

The first author was supported in part by NSF Grants DMS-07010001 and DMS-1001317. The second author was supported in part by NSF Grant DMS-0901620.

THEOREM 1.1. *Let $\rho \in C_0(\mathbb{R}^n)$, $\rho \geqslant 0$. The following three cases hold.*

(1) *If $2 < p < n$, then $I_{n-\alpha}(\rho)$ is p-superharmonic when*

$$0 < \alpha \leqslant \frac{n-p}{p-1}.$$

(2) *If $p > n$, then $I_{n-\alpha}(\rho)$ is p-subharmonic when*

$$-\alpha \geqslant \frac{p-n}{p-1}.$$

If $p = \infty$, one may take $-\alpha \geqslant 1$.

(3) *If $p = n$, then the function $I_n(\rho)(x) = \int_{\mathbb{R}^n} \rho(y) \log|x-y|\, dy$ is n-subharmonic.*

It is remarkable that the threshold $(n-p)/(p-1)$ is connected with the fundamental solution of the nonlinear operator Δ_p which, when $p \neq n$, is given by a multiple of the singular function $|x-y|^{-(n-p)/(p-1)}$. Theorem 1.1 was inspired by a recent paper of Crandall and Zhang [8] in which the authors establish a superposition principle stating that, for instance, sums like

$$\sum \frac{A_j}{|x-a_j|^{(n-p)/(p-1)}}, \quad A_j \geqslant 0,$$

are p-superharmonic in $\mathbb{R}^n$ when $2 < p < n$.

Because they arise as Euler–Lagrange equations for variational problems associated to best constants in Folland–Stein–Sobolev embedding theorems, in recent years there has been considerable study of nonlinear equations of p-Laplacian type in stratified nilpotent Lie groups. The basic prototype of such groups is the Heisenberg group $\mathbb{H}^n$ with its complex structure inherited from $\mathbb{C}^{n+1}$. In this model geometric setting, the p-Laplacian equation possesses remarkable explicit anisotropic fundamental solutions and there is nowadays a well-developed nonlinear potential theory associated with such singular kernels. In view of these facts, it seems natural to ask whether phenomena like those observed by Crandall–Zhang and Lindqvist–Manfredi have counterparts in the nonEuclidean setting of the Heisenberg group.

In this paper, we answer this question in the affirmative. In fact, more generally, we prove a superposition principle akin to Theorem 1.1 for a class of Riesz potentials which are naturally associated with sub-Laplacians on Lie groups of Heisenberg type. Such groups generalize the Heisenberg group and, as we explain in the course of the proof of our main result, they are presumably the largest nonEuclidean framework in which the Crandall–Zhang phenomenon holds. Given the intricate geometry associated with the complicated structure of such groups, it was for us surprising to discover that Theorem 1.1 extends to this significantly wider class of ambients. As the reader will see from a comparison of the proof in [17] with that of Theorem 1.3 in Section 4, the computations are considerably more complicated in the setting of this paper and it is only thanks to the special symmetries of groups of H-type that one is in the end able to control all quantities involved.

To state our main result, we recall that a Carnot group of step two is a connected, simply connected Lie group $\mathbb{G}$ whose Lie algebra admits a decomposition $\mathfrak{g} = V_1 \oplus V_2$ such that $[V_1, V_1] = V_2$ and $[V_1, V_2] = \{0\}$. We assume that $\mathfrak{g}$ has been endowed with an inner product $\langle \cdot, \cdot \rangle$ with respect to which the layers V_1 and V_2 are orthogonal. Let m be the dimension of V_1 and k be the dimension of V_2, and denote with $\{e_1, \dots, e_m\}$ and $\{\varepsilon_1, \dots, \varepsilon_k\}$ orthonormal bases of V_1 and V_2 with respect to $\langle \cdot, \cdot \rangle$. We always assume that both m and k are positive.

The left translation map $L_g : \mathbb{G} \to \mathbb{G}$ is defined by $L_g(g') = gg'$. Its differential will be indicated by dL_g. Using the above basis, we introduce left invariant vector fields by letting

$$X_i(g) = dL_g(e_i), \quad i = 1, \dots, m,$$
$$T_s(g) = dL_g(\varepsilon_s), \quad s = 1, \dots, k.$$

Hereafter, we agree that $\mathbb{G}$ is endowed with a left invariant Riemannian metric with respect to which the vector fields $X_1, \ldots, X_m, T_1, \ldots, T_k$ constitute an orthonormal basis.

The horizontal Laplacian (also known as sub-Laplacian) associated with the basis $\{e_1, \ldots, e_m\}$ is given by

$$\mathcal{L} = \sum_{i=1}^{m} X_i^2. \tag{1.1}$$

Observe that (1.1) fails to be elliptic at every point and that it is quite different from the (Riemannian) Laplace–Beltrami operator on $\mathbb{G}$, which is instead given by $\sum_{i=1}^{m} X_i^2 + \sum_{s=1}^{k} T_s^2$. However, thanks to the grading assumption $[V_1, V_1] = V_2$, the vector fields $X_1, \ldots, X_m$, together with their commutators, generate the full tangent bundle of $\mathbb{G}$, and therefore by Hörmander's theorem [**14**] the second-order differential operator $\mathcal{L}$ is hypoelliptic.

Since the exponential mapping $\exp : \mathfrak{g} \to \mathbb{G}$ is a surjective diffeomorphism, we can define a global system of coordinates on $\mathbb{G}$ as follows. Consider the analytic mappings $z : \mathbb{G} \to V_1$, $t : \mathbb{G} \to V_2$ defined through the equation $g = \exp(z(g) + t(g))$. For each $i = 1, \ldots, m$ we set

$$z_i = z_i(g) = \langle z(g), e_i \rangle,$$

whereas for $s = 1, \ldots, k$ we let

$$t_s = t_s(g) = \langle t(g), \varepsilon_s \rangle.$$

Henceforth, we will routinely omit the reference to the point $g \in \mathbb{G}$ and simply identify g with its logarithmic coordinates $\exp^{-1}(g) = z(g) + t(g)$, where $z(g) = z_1(g)e_1 + \ldots + z_m(g)e_m$, $t(g) = t_1(g)\varepsilon_1 + \ldots + t_k(g)\varepsilon_k(g)$. To simplify notation, we will also routinely identity $z \in V_1$ with the vector $(z_1, \ldots, z_m) \in \mathbb{R}^m$, and $t \in V_2$ with $(t_1, \ldots, t_k)$. For $g \in \mathbb{G}$ we write

$$g = (z, t) = (z_1, \ldots, z_m, t_1, \ldots, t_k).$$

The number

$$Q = m + 2k, \tag{1.2}$$

associated with the nonisotropic dilations $\Delta_\lambda(z + t) = \lambda z + \lambda^2 t$ on $\mathfrak{g}$, is called the homogeneous dimension of the group $\mathbb{G}$. It plays the role of a dimension in the analysis of $\mathbb{G}$. In fact, if we denote by $d\xi$ the standard $(m + k)$-dimensional Lebesgue measure on $\mathfrak{g}$, then one has $d(\Delta_\lambda(\xi)) = \lambda^Q \, d\xi$. Since a bi-invariant Haar measure on $\mathbb{G}$ is obtained by pushing forward $d\xi$ via the exponential mapping, such change of variable formula continues to be valid on $\mathbb{G}$ for the natural nonisotropic dilations defined by the formula

$$\delta_\lambda(g) = \exp \circ \Delta_\lambda \circ \exp^{-1}(g).$$

We will indicate with dg the Haar measure on $\mathbb{G}$. We thus have

$$d(\delta_\lambda(g)) = \lambda^Q \, dg.$$

We now consider the map $J : V_2 \to \mathrm{End}(V_1)$ which is uniquely specified by the identity

$$\langle J(t)z, z' \rangle = \langle [z, z'], t \rangle, \quad z, z' \in V_1, \ t \in V_2.$$

Definition 1.2. A Carnot group of step two $\mathbb{G}$ is said to be of *Heisenberg type*, or of H-type, if for every $t \in V_2$ with $|t| = 1$, the map $J(t)$ is an orthogonal endomorphism of V_1.

From Definition 1.2 we immediately see that, if $\mathbb{G}$ is of H-type, then

$$|J(t)z| = |z||t|, \quad z \in V_1, \ t \in V_2. \tag{1.3}$$

From (1.3) and the polarization identity, we obtain

$$\langle J(t)z, J(t')z\rangle = \langle t, t'\rangle |z|^2, \quad z \in V_1, t,\ t' \in V_2. \tag{1.4}$$

Groups of Heisenberg type were introduced by Kaplan [**15**] in connection with the hypoellipticity of sub-Laplacians. They constitute a natural generalization of the Heisenberg group $\mathbb{H}^n$, which (up to group isomorphism) one obtains from Definition 1.2 when the center V_2 is one-dimensional. For more information on the Heisenberg group, we refer the reader to [**19**] and to the monograph [**4**]. As it turns out, there is in nature a plentiful supply of H-type groups since, for instance, they arise in the Iwasawa decomposition of simple groups of rank 1; see [**7**].

The Folland–Kaplan gauge on $\mathbb{G}$ is defined by

$$N(g) = (|z|^4 + 16|t|^2)^{1/4}. \tag{1.5}$$

We note explicitly that $N(g) = N(g^{-1})$. Although such function can be defined in every Carnot group of step two, it has special significance if the group is of H-type. In such case, a remarkable discovery of Folland (for the Heisenberg group $\mathbb{H}^n$) [**10**], and Kaplan (for any H-type group) [**15**], indicates that the fundamental solution of the sub-Laplacian (1.1) is given by

$$\Gamma(g, g') = -c(\mathbb{G}) N(g^{-1}g')^{2-Q}, \tag{1.6}$$

where $c(\mathbb{G}) > 0$ is a suitable constant.

We now introduce the *Riesz potential of order* α on $\mathbb{G}$. Given $\rho \in C_0(\mathbb{G})$, we set

$$R_\alpha(\rho)(g) = \int_{\mathbb{G}} \frac{\rho(g')}{N(g^{-1}g')^{Q-\alpha}} dg', \quad 0 < \alpha < Q. \tag{1.7}$$

In view of (1.6), we see that, up to a negative constant, $R_2(\rho)$ coincides with $\Gamma \star \rho$, where we have indicated with $\star$ the group convolution in $\mathbb{G}$. But then, the potential identity $\mathcal{L}(\Gamma \star \rho) = \rho$, established in [**11**] for any Carnot group, allows to conclude that

$$-\mathcal{L}(R_2(\rho)) = \rho.$$

Hence, in a group of Heisenberg type $\mathbb{G}$ the Riesz potential R_2 plays the same role as the classical Newtonian potential I_2 in $\mathbb{R}^n$.

These considerations led us to inquire whether, in such groups, the 'linear' objects R_α could in any way be intertwined with a natural nonlinear operator on $\mathbb{G}$ which has received considerable attention over the past decade. Such operator presents itself in the Euler–Lagrange equation of the energy functional

$$E_p(\phi) = \frac{1}{p}\int_{\mathbb{G}} |X\phi|^p \, dg, \quad 1 < p < \infty,$$

in the Folland–Stein–Sobolev embedding; see [**11**]. Here, we indicated by $X\phi = \sum_{i=1}^m X_i\phi X_i$ the subgradient of a function ϕ and by $|X\phi| = (\sum_{i=1}^m (X_i\phi)^2)^{1/2}$ its length. An elementary calculation of the first variation of the energy E_p leads to the *horizontal p-Laplacian*, which is defined on $\mathbb{G}$ via its action on a smooth function ϕ by

$$\mathcal{L}_p\phi = \sum_{i=1}^m X_i(|X\phi|^{p-2}X_i\phi), \quad 1 < p < \infty. \tag{1.8}$$

Super- (sub-)solutions of the operator $\mathcal{L}_p$ must be defined in the weak sense, see [**5**], but at least for the range $p \geqslant 2$ of interest in this paper, when the function ϕ is sufficiently smooth, one can see that ϕ is p-superharmonic (p-subharmonic) if $\mathcal{L}_p\phi \leqslant 0$ ($\mathcal{L}_p\phi \geqslant 0$).

Returning to the aforementioned question of whether there is a result such as Theorem 1.1 in H-type groups, our answer is contained in the following theorem, which is the main result of the present paper.

THEOREM 1.3. *Let $\mathbb{G}$ be a group of Heisenberg type with homogeneous dimension Q given by (1.2). For a given $\rho \in C_0(\mathbb{G})$, $\rho \geqslant 0$, the following three cases hold.*

(1) *If $2 < p < Q$, then $R_{Q-\alpha}(\rho)$ is p-superharmonic when*

$$0 < \alpha \leqslant \frac{Q-p}{p-1}.$$

(2) *If $p > Q$, then $R_{Q-\alpha}(\rho)$ is p-subharmonic when*

$$-\alpha \geqslant \frac{p-Q}{p-1}.$$

If $p = \infty$, one may take $-\alpha \geqslant 1$.

(3) *If $p = Q$, then the function $R_Q(\rho)(g) = \int_{\mathbb{G}} \rho(g') \log N(g^{-1}g')\, dg'$ is Q-subharmonic.*

The reader should note the striking resemblance between Theorem 1.3 and its Euclidean predecessor Theorem 1.1. Again, a notable aspect here is the fact that the threshold exponent $(Q-p)/(p-1)$ is decided by that of the singular solutions of $\mathcal{L}_p$; see Proposition 3.1 in Section 3.

2. *Calculus on H-type groups*

Let $\mathbb{G}$ be a Carnot group of step two with its left invariant Riemannian tensor with respect to which the vector fields $X_1, \ldots, X_m, T_1, \ldots, T_k$ are orthonormal at every point. Given a smooth function $u : \mathbb{G} \to \mathbb{R}$, the subgradient (or horizontal gradient) of u is given by

$$Xu = \sum_{i=1}^{m} X_i u X_i.$$

Using the Baker–Campbell–Hausdorff formula, which for a Carnot group of step two asserts that

$$\exp(z+t)\exp(z'+t') = \exp(z + z' + t + t' + \tfrac{1}{2}[z, z']),$$

it is easy to see that in the logarithmic coordinates (z, t) one has

$$X_i = \partial_{z_i} - \frac{1}{2}\sum_{s=1}^{k}\sum_{j=1}^{m} b_{ij}^s z_j \partial_{t_s}, \tag{2.1}$$

where for $i, j = 1, \ldots, m,\ s = 1, \ldots, k$, we have indicated with

$$b_{ij}^s = \langle [e_i, e_j], \varepsilon_s \rangle = \langle J(\varepsilon_s) e_i, e_j \rangle, \tag{2.2}$$

the group constants. Note that for each $s = 1, \ldots, k$, the $m \times m$ matrix $[b_{ij}^s]$ is skew-symmetric, that is, $b_{ii}^s = 0$ and $b_{ji}^s = -b_{ij}^s$. We also introduce the convenient notation

$$B_{is} = B_{is}(g) = -\sum_{j=1}^{m} b_{ij}^s z_j = \langle J(\varepsilon_s) z, e_i \rangle. \tag{2.3}$$

From (2.1) it is not difficult to obtain the commutator formula

$$[X_i, X_j] = \sum_{s=1}^{k} b_{ij}^s \partial_{t_s}, \tag{2.4}$$

which shows, in particular, that in contrast with the Euclidean case the *horizontal Hessian*

$$X^2 u = (X_i X_j u)_{i,j=1,\ldots,m},$$

is not symmetric. More frequently, we will consider the *symmetrized horizontal Hessian*

$$(X^2u)^* = (u_{,ij})_{i,j=1,\dots,m},$$

where we have defined

$$u_{,ij} := \tfrac{1}{2}(X_iX_ju + X_jX_iu) = X_iX_j - \tfrac{1}{2}[X_i, X_j].$$

In the proof of Theorem 1.3, we will need the horizontal gradient and symmetrized horizontal Hessian of powers of the homogeneous norm N. In order to compute such quantities, we will build up to N through several preliminary steps. We introduce three more functions $\psi, \chi, a : \mathbb{G} \to \mathbb{R}$ by the formulas

$$\psi(g) := |z(g)|^2 = \sum_{i=1}^{m} z_i^2,$$

$$\chi(g) := |t(g)|^2 = \sum_{s=1}^{k} t_s^2$$

and

$$a(g) := \psi(g)^2 + 16\chi(g) = N(g)^4.$$

We will compute first the horizontal derivatives of ψ, then χ, then a and finally N.

Lemma 2.1. *Let $\mathbb{G}$ be a Carnot group of step two.*

(1) *For any i and j, $X_i(z_j) = \delta_{ij}$. Then $X_i\psi = 2z_i$ and $|X\psi|^2 = 4\psi$.*
(2) *For any i, j and ℓ, $X_iX_j(z_\ell) = 0$. Then $X_iX_j\psi = 2\delta_{ij}$ and $\psi_{,ij} = 2\delta_{ij}$.*

This lemma is completely trivial; the proof will be omitted. Using (2.1)–(2.3), it is not difficult to prove the following result.

Lemma 2.2. *Let $\mathbb{G}$ be a Carnot group of step two.*

(1) *For any i and s, we have $X_it_s = \frac{1}{2}B_{is}$. Then*

$$X_i\chi = \langle J(t)z, e_i\rangle,$$

and $|X\chi|^2 = |J(t)z)|^2$. In particular, if $\mathbb{G}$ is of H-type, we have from (1.3) $|X\chi|^2 = \psi\chi$.
(2) *For any $i, j = 1, \dots, m$ and $s = 1, \dots, k$, one has*

$$X_iX_jt_s = \tfrac{1}{2}\langle J(\varepsilon_s)e_i, e_j\rangle = \tfrac{1}{2}b^s_{ij}.$$

Then

$$X_iX_j\chi = \langle J(t)e_i, e_j\rangle + \frac{1}{2}\sum_s B_{is}B_{js}$$

and

$$\chi_{,ij} = \frac{1}{2}\sum_s B_{is}B_{js}.$$

This lemma is proved in [9] (see Proposition 6.5). We observe in passing some useful identities related to the coefficients B_{is}, which follow from the symmetry properties of the symplectic map J. First, for any $s = 1, \dots, k$, we have

$$\sum_i B_{is}z_i = \langle J(\varepsilon_s)z, z\rangle = 0. \tag{2.5}$$

Second, when $\mathbb{G}$ is of H-type, then using (1.4) for any $r, s = 1, \dots, k$, we obtain

$$\sum_i B_{ir} B_{is} = \langle J(\varepsilon_r)z, J(\varepsilon_s)z\rangle = \langle \varepsilon_r, \varepsilon_s\rangle |z|^2 = \delta_{rs}\psi. \tag{2.6}$$

The expression $|z|^2 z + 4J(t)z$ will appear repeatedly in what follows. We introduce a special notation for this expression, writing

$$A = \psi z + 4J(t)z. \tag{2.7}$$

Note that $\langle A, z\rangle = \psi|z|^2 = \psi^2$. When $\mathbb{G}$ is of H-type, then using (1.3) we find

$$|A|^2 = \psi^2|z|^2 + 16|z|^2|t|^2 = \psi a. \tag{2.8}$$

LEMMA 2.3. *Let $\mathbb{G}$ be a Carnot group of step two, then $Xa = 4A$. Moreover, for any $i, j = 1, \dots, m$,*

$$a_{,ij} = 4\psi\delta_{ij} + 8\left(z_i z_j + \sum_s B_{is} B_{js}\right).$$

In particular, when $\mathbb{G}$ is of H-type, then $|Xa|^2 = 16\psi a$.

Both parts of the lemma are direct computations using the formulas for the derivatives of ψ and χ in the preceding lemmas, together with the formula for a.

Finally, we arrive at the formulas for the horizontal gradient and symmetrized horizontal Hessian of the homogeneous norm N.

LEMMA 2.4. *Let $\mathbb{G}$ be a Carnot group of step two, then*

$$XN = N^{-3}A.$$

Furthermore, for any $i, j = 1, \dots, m$,

$$N_{,ij} = N^{-7}\left(a\psi\delta_{ij} + 2a\left(z_i z_j + \sum_s B_{is}B_{js}\right) - 3\langle A, e_i\rangle\langle A, e_j\rangle\right).$$

When $\mathbb{G}$ is of H-type, then

$$|XN|^2 = N^{-2}\psi.$$

The first part of Lemma 2.4 is an immediate consequence of Lemma 2.3 and the identities $X_i N = \frac{1}{4}a^{-3/4}X_i a$ and

$$N_{,ij} = \tfrac{1}{4}a^{-3/4}a_{,ij} - \tfrac{3}{16}a^{-7/4}X_i a X_j a.$$

The second part follows directly from (2.8).

3. *Singular p-harmonic functions*

The basic objects of study in this paper are the horizontal Laplacian (1.1) and its nonlinear generalization, the horizontal p-Laplacian (1.8). We also consider the *horizontal ∞-Laplacian*

$$\mathcal{L}_\infty u = \sum_{i,j} u_{,ij} X_i u X_j u = \langle (X^2 u)^* Xu, Xu\rangle,$$

which arises formally as a term in the limit of $(1/p)\mathcal{L}_p u$ as $p \to \infty$; see (3.1).

These nonlinear operators are only well-defined on functions that are twice horizontally differentiable (and even on such functions, the action of $\mathcal{L}_p$ in the range $1 < p < 2$ is not completely justified). The correct notion of p-superharmonic (or p-subharmonic) function must be introduced in the standard weak sense; see [**5**]. In this paper, however, we apply the operator $\mathcal{L}_p$ only to functions that are sufficiently smooth, and only in the range $p > 2$. Therefore, we will perform all our computations using strong derivatives. In this framework, we say that u is *p-subharmonic* if $\mathcal{L}_p u \geqslant 0$. Similarly, we say that u is *p-superharmonic* if $\mathcal{L}_p u \leqslant 0$. We also have the following formula

$$\mathcal{L}_p u = |Xu|^{p-2}\mathcal{L}u + (p-2)|Xu|^{p-4}\mathcal{L}_\infty u, \tag{3.1}$$

which relates the horizontal p-Laplacian to the horizontal Laplacian and ∞-Laplacian.

Let us remark in passing that the equivalence of viscosity and weak solutions to the p-Laplacian in Euclidean space provided significant motivation for the original investigations of the superposition principle for p-superharmonic functions by Crandall–Zhang and Lindqvist–Manfredi. On the Heisenberg group, the equivalence of the corresponding notions of viscosity and weak solutions to the p-Laplace equation is due to Bieske [**2**].

As we have mentioned in Section 1, one remarkable aspect of H-type groups is the fact that the fundamental solution of the horizontal Laplacian is given by the formula (1.6). Interestingly, this phenomenon continues to hold in the nonlinear case. One has in fact the following result from [**6**] (see also [**12**] for the case $p = Q$).

Proposition 3.1. *For every $1 < p < \infty$ the function*

$$\Gamma_p(g, g') = \Gamma_p(g', g) = \begin{cases} -\dfrac{p-1}{Q-p}\sigma_p^{-1/(p-1)} N(g^{-1}g')^{(p-Q)/(p-1)}, & p \neq Q, \\ -\sigma_p^{-1/(p-1)} \log N(g^{-1}g'), & p = Q, \end{cases} \tag{3.2}$$

with $g' \neq g$, is a fundamental solution of (1.8) with singularity at $g \in \mathbb{G}$.

In (3.2), we have let $\sigma_p = Q\omega_p$, where

$$\omega_p = \int_{B(e,1)} |XN(g)|^p \, dg.$$

The meaning of Proposition 3.1 is that for every $\phi \in C_0^\infty(\mathbb{G})$ one has

$$\phi(g) = \int_{\mathbb{G}} |X\Gamma_p(g, g')|^{p-2} \langle X\Gamma_p(g, g'), X\phi(g')\rangle \, dg'.$$

The fundamental solution $\Gamma(g, \cdot)$ is C^∞ smooth in the domain $\mathbb{G} \setminus \{g\}$, where it satisfies

$$\mathcal{L}_p(\Gamma_p(g, \cdot)) \equiv 0.$$

Note that for every $g \neq g'$ we have

$$\lim_{p\to\infty} \Gamma_p(g, g') = N(g^{-1}g').$$

This suggests that we should also have

$$\mathcal{L}_\infty(N(g^{-1}\cdot)) \equiv 0 \quad \text{in } \mathbb{G} \setminus \{g\}. \tag{3.3}$$

The identity (3.3) is also true; see [**9**, Proposition 6.4]. In fact, the homogeneous norm N is a viscosity solution of the horizontal ∞-Laplacian; see [**1**, **3**, **20**].

4. *Proof of Theorem* 1.3

Let α and ρ be as in the statement of the theorem, and define

$$F(g) = R_{Q-\alpha}(\rho)(g) = \int_{\mathbb{G}} \frac{\rho(g')}{N((g')^{-1}g)^{\alpha}} dg' \quad \text{if } 0 \neq \alpha < Q \tag{4.1}$$

and

$$F(g) = R_Q(\rho)(g) = \int_{\mathbb{G}} \rho(g') \log N(g^{-1}g')dg' \quad \text{if } \alpha = 0.$$

Here we used the fact that $N((g')^{-1}g) = N(g^{-1}g')$.

Let $2 < p < \infty$; the case $p = \infty$ can be seen as a suitable limit of the following computations. Using (3.1), we have

$$|XF|^{4-p}\mathcal{L}_pF = |XF|^2\mathcal{L}F + (p-2)\mathcal{L}_\infty F. \tag{4.2}$$

If we now set $q = -\alpha$, then differentiating under the integral sign, and using the left-invariance of the vector fields $X_1, \ldots, X_m$, we find that

$$X_iF(g) = q\int_{\mathbb{G}} N((g')^{-1}g)^{q-1}X_iN((g')^{-1}g)\rho(g')\,dg' \tag{4.3}$$

and also that

$$\begin{aligned} F_{,ij}(g) &= q\int_{\mathbb{G}} N((g')^{-1}g)^{q-1}N_{,ij}((g')^{-1}g)\rho(g')\,dg' \\ &\quad + q(q-1)\int_{\mathbb{G}} N((g')^{-1}g)^{q-2}X_iN((g')^{-1}g)X_jN((g')^{-1}g)\rho(g')\,dg', \end{aligned} \tag{4.4}$$

if $q \neq 0$. (In the case $q = -\alpha = 0$, the formulas must be modified by removing the various factors of q which appear in (4.3) and (4.4).)

Before proceeding any further, we introduce another simplifying notation. In the remainder of the proof, we will have need for all of the data computed in the preceding lemmas, evaluated at different places in the group, for example, at $(g')^{-1}g$ for various points g'. In order to keep the formulas involved from expanding out of control, we will write

$$N_{g'}(g) := N((g')^{-1}g).$$

After factoring out a constant multiple of q^3 from all terms, the right-hand side of (4.2) is equal to the sum, over i and j, of the expression

$$\begin{aligned} &\int_{\mathbb{G}} ((q-1)N_{g'}^{q-2}(X_iN_{g'})^2 + N_{g'}^{q-1}(N_{g'})_{,ii})\rho(g')\,dg' \left(\int_{\mathbb{G}} N_{g'}^{q-1}X_jN_{g'}\rho(g')\,dg'\right)^2 \\ &\quad + (p-2)\left(\int_{\mathbb{G}} ((q-1)N_{g'}^{q-2}X_iN_{g'}X_jN_{g'} + N_{g'}^{q-1}(N_{g'})_{,ij})\rho(g')\,dg'\right) \\ &\quad \times \left(\int_{\mathbb{G}} N_{g'}^{q-1}X_iN_{g'}\rho(g')\,dg'\right)\left(\int_{\mathbb{G}} N_{g'}^{q-1}X_jN_{g'}\rho(g')\,dg'\right). \end{aligned} \tag{4.5}$$

Recall that there is an implicit variable g in this expression: it is the argument of the various terms involving the norm. Our claim is that (4.5), as a function of g, has a definite sign on all of $\mathbb{G}$, depending only on the relative sizes of p and $q = -\alpha$ as indicated in the hypotheses.

We rewrite (4.5) as a triple integral, introducing dummy variables g_1', g_2' and g_3'. At this point, we will simplify the notation even further by abbreviating $N_1 := N_{g_1'}$, $\rho_1 := \rho(g_1')$ and so on. Then the right-hand side of (4.2) is equal to

$$q^3\sum_{i,j}\iiint \rho_1\rho_2\rho_3N_1^{q-2}N_2^{q-1}N_3^{q-1} \times \mathbf{M}_{ij}, \tag{4.6}$$

where

$$\begin{aligned}\mathbf{M}_{ij} &= (q-1)(X_iN_1)^2(X_jN_2)(X_jN_3) + N_1((N_1)_{,ii})(X_jN_2)(X_jN_3)\\ &\quad + (p-2)(q-1)(X_iN_1)(X_jN_1)(X_iN_2)(X_jN_3) + (p-2)N_1((N_1)_{,ij})(X_iN_2)(X_jN_3).\end{aligned}$$

Here, the integral is taken with respect to the three variables g_1', g_2', g_3', over $\mathbb{G}\times\mathbb{G}\times\mathbb{G}$.

Our next task is to substitute the expressions from Lemma 2.4 into the above formula. The ensuing equation is a monster expression. In order to gain some insight into the structure of the resulting formula, we separate it into two parts which we treat independently. We write

$$\mathbf{M}_{ij} = (q-1)\mathbf{M}^1_{ij} + \mathbf{M}^2_{ij},$$

where

$$\mathbf{M}^1_{ij} = (X_iN_1)^2(X_jN_2)(X_jN_3) + (p-2)(X_iN_1)(X_jN_1)(X_iN_2)(X_jN_3)$$

and

$$\mathbf{M}^2_{ij} = N_1((N_1)_{,ii})(X_jN_2)(X_jN_3) + (p-2)N_1((N_1)_{,ij})(X_iN_2)(X_jN_3).$$

Using Lemma 2.4 we compute

$$\mathbf{M}^1_{ij} = N_1^{-6}N_2^{-3}N_3^{-3}[\langle A_1,e_i\rangle^2\langle A_2,e_j\rangle\langle A_3,e_j\rangle + (p-2)\langle A_1,e_i\rangle\langle A_1,e_j\rangle\langle A_2,e_i\rangle\langle A_3,e_j\rangle]$$

and

$$\begin{aligned}\mathbf{M}^2_{ij} &= N_1^{-6}N_2^{-3}N_3^{-3}\\ &\quad\times\Bigg[\left(a_1\psi_1 + 2a_1(z_1)_i^2 + 2a_1\sum_s (B_1)^2_{is} - 3\langle A_1,e_i\rangle^2\right)\langle A_2,e_j\rangle\langle A_3,e_j\rangle\\ &\quad + (p-2)\Bigg(a_1\psi_1\delta_{ij} + 2a_1(z_1)_i(z_1)_j\\ &\quad + 2a_1\sum_s (B_1)_{is}(B_1)_{js} - 3\langle A_1,e_i\rangle\langle A_1,e_j\rangle\Bigg)\langle A_2,e_i\rangle\langle A_3,e_j\rangle\Bigg].\end{aligned}$$

Here, we have continued to use subscripts to denote the result of evaluation of the various functions a, ψ, A etc. at the points $(g')^{-1}g$, that is, $a_1 := a((g_1')^{-1}g)$ and so on. However, one notation might unfortunately generate confusion. The reader should pay attention to the fact that henceforth z_1 means $z((g_1')^{-1}g)$, not the first component of the vector $z = (z_1,\dots,z_m)$. Accordingly, the notation $(z_1)_i$ indicates the ith component of the vector $z((g_1')^{-1}g) \in V_1$.

Returning to (4.6), we sum in i and j. We make use of the fact that the vectors e_i form an orthonormal basis for V_1, and that the functions A_1, A_2, A_3 take values in V_1. Consequently, we obtain that the right-hand side of (4.2) is equal to the sum of the following two terms (which are obtained by summing the expressions for $(q-1)\mathbf{M}^1_{ij}$ and $\mathbf{M}^2_{ij}$, respectively):

$$q^3(q-1)\iiint \rho_1\rho_2\rho_3 N_1^{q-8}N_2^{q-4}N_3^{q-4}(|A_1|^2\langle A_2,A_3\rangle + (p-2)\langle A_1,A_2\rangle\langle A_1,A_3\rangle) \tag{4.7}$$

and

$$q^3 \iiint \rho_1\rho_2\rho_3 N_1^{q-8} N_2^{q-4} N_3^{q-4}$$
$$\times \Bigg[\Bigg((m+2)a_1\psi_1 + 2a_1 \sum_{i,s} (B_1)_{is}^2 - 3|A_1|^2 \Bigg) \langle A_2, A_3\rangle$$
$$+ (p-2) \Bigg(a_1\psi_1 \langle A_2, A_3\rangle + 2a_1 \langle A_2, z_1\rangle\langle A_3, z_1\rangle$$
$$+2a_1 \sum_s \langle A_2, J(\varepsilon_s)z_1\rangle\langle A_3, J(\varepsilon_s)z_1\rangle - 3\langle A_1, A_2\rangle\langle A_1, A_3\rangle \Bigg)\Bigg]. \tag{4.8}$$

(Recall that $m = \dim V_1$.) In the preceding, we also used the identity

$$\sum_{i,j} (B_1)_{is}\langle A_2, e_i\rangle (B_1)_{js}\langle A_3, e_j\rangle = \langle A_2, J(\varepsilon_s)z_1\rangle\langle A_3, J(\varepsilon_s)z_1\rangle, \tag{4.9}$$

valid for each $s = 1, \dots, k$. Identity (4.9) can be easily verified from (2.3) and from the orthonormality of $\{e_1, \dots, e_m\}$. By (2.6), $\sum_i (B_1)_{is}^2 = \psi$ for each $s = 1, \dots, k$. Hence $\sum_{i,s}(B_1)_{is}^2 = k\psi$, where k denotes the dimension of the second layer V_2.

Using (1.2) and (2.8), we regroup the various terms in (4.8) to obtain

$$q^3 \iiint \rho_1\rho_2\rho_3 N_1^{q-8} N_2^{q-4} N_3^{q-4}$$
$$\times \Bigg[(Q+p-3)|A_1|^2\langle A_2, A_3\rangle + 2(p-2)a_1\langle A_2, z_1\rangle\langle A_3, z_1\rangle$$
$$+2(p-2)a_1 \sum_s \langle A_2, J(\varepsilon_s)z_1\rangle\langle A_3, J(\varepsilon_s)z_1\rangle - 3(p-2)\langle A_1, A_2\rangle\langle A_1, A_3\rangle \Bigg]. \tag{4.10}$$

We now recombine (4.7) and (4.10) to obtain the following formula for the right-hand side of (4.2):

$$q^3 \iiint \rho_1\rho_2\rho_3 N_1^{q-8} N_2^{q-4} N_3^{q-4}$$
$$\times \Bigg[(Q+p+q-4)|A_1|^2\langle A_2, A_3\rangle + 2(p-2)a_1\langle A_2, z_1\rangle\langle A_3, z_1\rangle$$
$$+2(p-2)a_1 \sum_s \langle A_2, J(\varepsilon_s)z_1\rangle\langle A_3, J(\varepsilon_s)z_1\rangle + (q-4)(p-2)\langle A_1, A_2\rangle\langle A_1, A_3\rangle \Bigg]. \tag{4.11}$$

Observe that the dependence on the g_2' and g_3' variables occurs only in the expressions A_2 and A_3, and that the integrand is bilinear in those expressions. It is thus natural to introduce the new function

$$\mathcal{K}(g) = \int_{\mathbb{G}} \rho(g') N_{g'}^{q-4}(g) A_{g'}(g)\, dg'$$

obtained by integrating $A_{g'}$ against the coefficient $\rho(g')N_{g'}^{q-4}$ which appears in (4.11). Performing the integration in the g_2' and g_3' variables simplifies (4.11) as follows:

$$q^3\int \rho_1 N_1^{q-8}\Bigg[(Q+p+q-4)|A_1|^2|\mathcal{K}|^2+(q-2)(p-2)\langle A_1,\mathcal{K}\rangle^2 \\ +2(p-2)\left(a_1\langle \mathcal{K},z_1\rangle^2+a_1\sum_s\langle \mathcal{K},J(\varepsilon_s)z_1\rangle^2-\langle A_1,\mathcal{K}\rangle^2\right)\Bigg]. \tag{4.12}$$

Observe that

$$(Q+p+q-4)|A_1|^2|\mathcal{K}|^2+(q-2)(p-2)\langle A_1,\mathcal{K}\rangle^2 \geqslant 0,$$

for all A_1 and $\mathcal{K}$ by the Cauchy–Schwarz inequality, provided that

$$q=-\alpha \geqslant \frac{p-Q}{p-1}$$

precisely as hypothesized. To complete the proof, we will show that the expression

$$a_1\langle \mathcal{K},z_1\rangle^2+a_1\sum_s\langle \mathcal{K},J(\varepsilon_s)z_1\rangle^2-\langle A_1,\mathcal{K}\rangle^2 \tag{4.13}$$

is always nonnegative. Using (2.7), we compute $\langle A_1,\mathcal{K}\rangle=\psi_1\langle\mathcal{K},z_1\rangle+4\langle\mathcal{K},J(t_1)z_1\rangle$ where, as above for z_1, the notation t_1 indicates $t((g_1')^{-1}g)$, and not the first component of the vector $t=(t_1,\dots,t_k)$. Since $a=\psi^2+16\chi$, the quantity in (4.13) is equal to

$$16\chi_1\langle\mathcal{K},z_1\rangle^2-8\psi_1\langle\mathcal{K},z_1\rangle\langle\mathcal{K},J(t_1)z_1\rangle-16\langle\mathcal{K},J(t_1)z_1\rangle^2+((\psi_1)^2+16\chi_1)\sum_s\langle\mathcal{K},J(\varepsilon_s)z_1\rangle^2,$$

which we write as the sum of the following two expressions:

$$\mathrm{I}:=16\chi_1\langle\mathcal{K},z_1\rangle^2-8\psi_1\langle\mathcal{K},z_1\rangle\langle\mathcal{K},J(t_1)z_1\rangle+(\psi_1)^2\sum_s\langle\mathcal{K},J(\varepsilon_s)z_1\rangle^2$$

and

$$\mathrm{II}:=16\chi_1\sum_s\langle\mathcal{K},J(\varepsilon_s)z_1\rangle^2-16\langle\mathcal{K},J(t_1)z_1\rangle^2.$$

The proof is now finished by observing that

$$\mathrm{I}=\sum_s(4(t_s)_1\langle\mathcal{K},z_1\rangle-\psi_1\langle\mathcal{K},J(\varepsilon_s)z_1\rangle)^2,$$

while II is nonnegative by another application of the Cauchy–Schwarz inequality. The assertion about I follows by recognizing that

$$\sum_s t_s\langle\mathcal{K},J(\varepsilon_s)z\rangle=\sum_s\langle t,\varepsilon_s\rangle\langle[z,\mathcal{K}],\varepsilon_s\rangle=\langle t,[z,\mathcal{K}]\rangle=\langle\mathcal{K},J(t)z\rangle.$$

In conclusion, we have shown that the integral in (4.12) is nonnegative, under the stated assumptions on p and $q=-\alpha$. The factor of q^3 then ensures that the p-Laplacian takes on the appropriate sign (based on whether $p<Q$ or $p>Q$). Note that q is strictly negative in case (1) of the theorem, while q is strictly positive in case (2). In case (3) ($p=Q$) the argument is identical, except that the overall factor of q in (4.3) and (4.4) should be dropped. After that, setting $q=0$ and repeating the argument yields the desired conclusion. This completes the proof of the theorem.

Remark 4.1. In closing, we mention that Theorem 1.3 can be generalized to the case of Riesz potentials

$$R_\alpha(\mu)(g) = \int_{\mathbb{G}} \frac{d\mu(g')}{N(g^{-1}g')^{Q-\alpha}}$$

of rather general Radon measures μ. If we consider a Radon measure μ on $\mathbb{G}$, then, similarly to what has been done in [**17**], we can show that Theorem 1.3 continues to be valid for the Riesz potential $R_{Q-\alpha}(\mu)$, provided μ fulfills the growth assumption

$$\int_{\{N(g)\geqslant 1\}} \frac{d\mu(g)}{N(g)^\alpha} < \infty.$$

We omit the details.

Acknowledgements. Research for this paper was completed while the authors were visitors in the Intensive Research Period 'Euclidean Harmonic Analysis, Nilpotent Lie groups and PDE' at the Centro di Ricerca Matematica Ennio De Giorgi in April 2010. We are grateful to the De Giorgi Center for the warm hospitality and the excellent working atmosphere. We thank the referee for a careful reading of the paper and for helpful comments.

References

1. T. Bieske, 'On ∞-harmonic functions on the Heisenberg group', *Comm. Partial Differential Equations* 27 (2002) 727–761.

2. T. Bieske, 'Equivalence of weak and viscosity solutions to the p-Laplace equation on the Heisenberg group', *Ann. Acad. Sci. Fenn. Ser. A I Math.* 31 (2006) 363–379.

3. T. Bieske and L. Capogna, 'The Aronsson–Euler equation for absolutely minimizing Lipschitz extensions with respect to Carnot–Carathéodory metrics', *Trans. Amer. Math. Soc.* 357 (2005) 795–823.

4. A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer Monographs in Mathematics (Springer, Berlin, 2007).

5. L. Capogna, D. Danielli and N. Garofalo, 'An embedding theorem and the Harnack inequality for nonlinear subelliptic equations', *Comm. Partial Differential Equations* 18 (1993) 1765–1794.

6. L. Capogna, D. Danielli and N. Garofalo, 'Capacitary estimates and the local behavior of solutions of nonlinear subelliptic equations', *Amer. J. Math.* 118 (1996) 1153–1196.

7. M. Cowling, A. H. Dooley, A. Korányi and F. Ricci, 'H-type groups and Iwasawa decompositions', *Adv. Math.* 87 (1991) 1–41.

8. M. G. Crandall and J. Zhang, 'Another way to say harmonic', *Trans. Amer. Math. Soc.* 355 (2003) 241–263.

9. D. Danielli, N. Garofalo and D. M. Nhieu, 'Notions of convexity in Carnot groups', *Comm. Anal. Geom.* 11 (2003) 263–341.

10. G. B. Folland, 'A fundamental solution for a subelliptic operator', *Bull. Amer. Math. Soc.* 79 (1973) 373–376.

11. G. B. Folland, 'Subelliptic estimates and function spaces on nilpotent Lie groups', *Ark. Mat.* 13 (1975) 161–207.

12. J. Heinonen and I. Holopainen, 'Quasiregular maps on Carnot groups', *J. Geom. Anal.* 7 (1997) 109–148.

13. L. L. Helms, *Potential theory*, Universitext (Springer, London, 2009).

14. L. Hörmander, 'Hypoelliptic second order differential equations', *Acta Math.* 119 (1967) 147–171.

15. A. Kaplan, 'Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms', *Trans. Amer. Math. Soc.* 258 (1980) 147–153.

16. N. S. Landkof, *Foundations of modern potential theory*, Grundlehren der mathematischen Wissenschaften 180 (Springer, New York, 1972).

17. P. Lindqvist and J. J. Manfredi, 'Note on a remarkable superposition for a nonlinear equation', *Proc. Amer. Math. Soc.* 136 (2008) 133–140.

18. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series 30 (Princeton University Press, Princeton, NJ, 1970).

19. E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series 43, Monographs in Harmonic Analysis III (Princeton University Press, Princeton, NJ, 1993).

20. C. Wang, 'The Aronsson equation for absolute minimizers of L^∞-functionals associated with vector fields satisfying Hörmander's condition', *Trans. Amer. Math. Soc.* 359 (2007) 91–113.

Nicola Garofalo
Department of Mathematics
Purdue University
150 N. University St.
West Lafayette, IN 47907
USA

garofalo@math.purdue.edu

Jeremy T. Tyson
Department of Mathematics
University of Illinois at Urbana-Champaign
1409 West Green St.
Urbana, IL 61801
USA

tyson@math.uiuc.edu