On the Falk invariant of signed graphic arrangements

Weili Guo (joint with Michele Torielli)

(arXiv: 1703.09402)

Department of Mathematics, Hokkaido University

Introduction

Let $M := \mathbb{C}^\ell \setminus_{H\in\mathcal{A}} H$ be the complement of the arrangement $\mathcal{A}$. It is known that the cohomology ring $H^*(M)$ is completely determined by $L(\mathcal{A})$ the lattice of intersection of $\mathcal{A}$. Similarly to this result, there are several conjectures concerning the relationship between M and $L(\mathcal{A})$. To study such problems, Falk([1][2]) introduced in a multiplicative invariant, called **global invariant**, of the Orlik-Solomon algebra of $\mathcal{A}$. The invariant is now known as the (3^{rd}) **Falk invariant** and it is denoted by ϕ_3. In [2], Falk posed as an open problem to give a combinatorial interpretation of ϕ_3. In [3], Schenck and Suciu studied the lower central series of arrangements and described a formula for the Falk invariant in the case of graphic arrangements.

Our main result is devoted to describe a combinatorial formula for the Falk invariant of a signed graphic arrangement that do not have a B_2 as sub-arrangement.

Notations

Orlik-Solomon algebras

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$. Let $E^1 = \bigoplus_{j=1}^n \mathbb{C}e_j$ be the free module generated by $e_1, e_2, \dots, e_n$, where e_i is a symbol corresponding to the hyperplane H_i. Let $E = \bigwedge E^1$ be the exterior algebra over $\mathbb{C}$. The algebra E is graded via $E = \bigoplus_{p=0}^n E^p$, where $E^p = \bigwedge^p E^1$. The $\mathbb{C}$-module E^p is free and has the distinguished basis consisting of monomials $e_S := e_{i_1} \wedge \cdots \wedge e_{i_p}$, where $S = \{i_1, \dots, i_p\}$ is running through all the subsets of $\{1, \dots, n\}$ of cardinality p and $i_1 < i_2 < \cdots < i_p$. The graded algebra E is a commutative DGA with respect to the differential ∂ of degree -1 uniquely defined by the conditions $\partial e_i = 1$ for all $i = 1, \dots, n$ and the graded Leibniz formula. Then for every $S \subseteq \{1, \dots, n\}$ of cardinality p

$$\partial e_S = \sum_{j=1}^{p} (-1)^{j-1} e_{S_j},$$

where S_j is the complement in S to its j-th element.

For every $S \subseteq \{1, \dots, n\}$, put $\cap S = \bigcap_{i\in S} H_i$ (possibly $\cap S = \emptyset$). The set of all intersections $L(\mathcal{A}) := \{\cap S \mid S \subseteq \{1, \dots, n\}\}$ is called the **intersection poset of** $\mathcal{A}$. The subset $S \subseteq \{1, \dots, n\}$ is called **dependent** if $\cap S \neq \emptyset$ and the set of linear polynomials $\{\alpha_i \mid i \in S\}$ with $H_i = \alpha_i^{-1}(0)$, is linearly dependent.

Definition 1. The **Orlik-Solomon ideal** of $\mathcal{A}$ is the ideal $I = I(\mathcal{A})$ of E generated by all e_S with $\cap S = \emptyset$, and all ∂e_S with S dependent. The algebra $A := A^\bullet(\mathcal{A}) = E/I(\mathcal{A})$ is called the **Orlik-Solomon algebra** of $\mathcal{A}$.

Let I_k be the k-adic Orlik-Solomon ideal of $\mathcal{A}$ generated by $\sum_{j\le k} I^j$ in E. It is clear that I^k is a graded ideal and $I_k^p = (I_k)^p = E^p \cap I_k$. Write $A_k := A_k^\bullet(\mathcal{A}) = E/I_k$ and $A_k^p := (A_k^\bullet(\mathcal{A}))^p = E^p/I_k^p$ which is called k**-adic Orlik-Solomon algebra** by Falk.

Theorem 2 (Falk,2001[2])**.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$. Then

$$\phi_3 = 2\binom{n+1}{3} - n\dim(A^2) + \dim(A_2^3). \tag{1}$$

Sign graphs

Definition 3. A **signed graph** is a tuple $G = (V_G, E_G^+, E_G^-, L_G)$, where

- V_G is a finite set called the set of vertices,
- E_G^+ is a subset of $\binom{V_G}{2}$ called the set of positive edges,
- E_G^- is a subset of $\binom{V_G}{2}$ called the set of negative edges,
- L_G is a subset of V_G called the set of loops.

Definition 4. Given a signed graph G, let $\mathcal{A}(G)$ be the hyperplane arrangement in $\mathbb{C}^\ell$ consisting of the following hyperplane

$$\{x_i - x_j = 0\} \text{ for } \{i,j\} \in E_G^+,$$

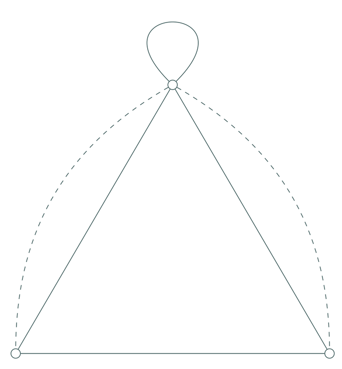
$$\{x_i + x_j = 0\} \text{ for } \{i,j\} \in E_G^-,$$

$$\{x_i = 0\} \text{ for } i \in L_G.$$

We will call $\mathcal{A}(G)$ the **signed graphic arrangement** associated to the signed graph G.

Corollary 5. Let G_1 and G_2 be two signed graph with the same underlying graph. If G_1 is switching equivalent to G_2, then $\phi_3(\mathcal{A}(G_1)) = \phi_3(\mathcal{A}(G_2))$.

Taking inspiration from graph theory and the study of hyperplane arrangements, we denote by K_ℓ a complete graph with ℓ vertices and all edges being positive, i.e. $K_\ell = (K_\ell, K_\ell^\circ, \emptyset)$, by D_ℓ a complete sign graph with ℓ vertices and no loops, i.e. $D_\ell = (K_\ell, K_\ell, \emptyset)$, and by B_ℓ a sign complete graph with ℓ vertices and a full set of loops, i.e. $B_\ell = (K_\ell, K_\ell, [\ell])$. Moreover, we denote by K_ℓ^ℓ a complete graph with ℓ vertices, all edges being positive and a full set of loops, i.e. $K_\ell^\ell = (K_\ell, K_\ell^\circ, [\ell])$, by D_ℓ^1 a complete sign graph with ℓ vertices and one loop, i.e. $D_\ell^1 = (K_\ell, K_\ell, \{1\})$ and by $G_\circ$ the signed graph in Figure 2. Furthermore, if G is a signed graph we denote but $\overline{G}$ a signed graph switching equivalent to G for some switching function σ.



(2)

Main Theorem

Our main theorem is the following,

Theorem 6. For a signed graphic arrangement associated to a signed graph G not containing a subgraph isomorphic to B_2 as subgraph, we have

$$\phi_3 = 2(k_3 + k_4 + d_3 + d_{2,1} + k_{2,2} + k_{3,3} + g_\circ) + 5d_{3,1}, \tag{3}$$

where k_l denotes the number of subgraph of G isomorphic to a $\overline{K_l}$, d_l denotes the number of subgraph of G isomorphic to D_l but not contained in D_l^1, $d_{l,1}$ denotes the number of subgraph of G isomorphic to D_l^1, $k_{l,l}$ denotes the number of subgraph of G isomorphic to a $\overline{K_l^l}$ and $g_\circ$ denotes the number of subgraph of G isomorphic to a $\overline{G_\circ}$ but not contained in D_l^1.

Proof of Main Theorem and Lemmas

Outline of proof

- **Step 1.** We apply the following lemma to compute $\dim(A^2)$.

Lemma 7. $\dim(A^2) = \binom{n}{2} - k_3 - d_{2,1} - k_{2,2}$.

- **Step 2.** We use the following lemmas to compute $\dim(I_2^3)$, then we can get $\dim(A_2^3)$.

With an abuse of notation, we will call a dependent 3-tuple S a **triangle**. Moreover, we will write

$$\mathcal{C}_3 := \{e_S \in E \mid S \text{ is a triangle}\}$$

which is a subset of E as a vector space over $\mathbb{C}$.Since $e_ie_je_k = -e_je_ie_k$, it is clear that the dimension of the vector space $\mathcal{C}_3$ is $k_3 + d_{2,1} + k_{2,2}$. Moreover, we can consider C_3' a basis of $\mathcal{C}_3$. Then each element of C_3' is in a one-to-one correspondence of the subgraph of G isomorphic to a $\overline{K_3}$, or a D_2^1 or a $\overline{K_2^2}$.

Write

$$C_3 := \{e_t\partial e_{ijk} \mid e_{ijk} \in C_3', t \in \{i,j,k\}\},$$

and

$$F_3 := \{e_t\partial e_{ijk} \mid e_{ijk} \in C_3', t \in [n] \setminus \{i,j,k\}\}.$$

Lemma 8. *For a signed graphic arrangement associated to a signed graph G not containing a subgraph isomorphic to B_2 as subgraph, we have*

$$I_2^3 = \mathrm{span}(C_3) \oplus \mathrm{span}(F_3).$$

To prove our main result we need to be able to compute $\dim(\mathrm{span}(F_3))$. To do so, consider the following sets

- $F_3^1 := \{e_t\partial e_{ijk} \mid e_{ijk} \in C_3', t \in [n] \setminus \{i,j,k\}, i,j,k$ are not in the same $\overline{K_4}, D_3, \overline{G_\circ}, D_3^1, \overline{K_3^3}\}$,
- $F_3^2 := \{e_t\partial e_{ijk} \mid e_{ijk} \in C_3', t \in [n] \setminus \{i,j,k\}, i,j,k$ are in the same $\overline{K_4}\}$,
- $F_3^3 := \{e_t\partial e_{ijk} \mid e_{ijk} \in C_3', t \in [n] \setminus \{i,j,k\}, i,j,k$ are in the same D_3 but not same $D_3^1\}$,
- $F_3^4 := \{e_t\partial e_{ijk} \mid e_{ijk} \in C_3', t \in [n] \setminus \{i,j,k\}, i,j,k$ are in the same $\overline{G_\circ}$ but not same $D_3^1\}$,
- $F_3^5 := \{e_t\partial e_{ijk} \mid e_{ijk} \in C_3', t \in [n] \setminus \{i,j,k\}, i,j,k$ are in the same $D_3^1\}$,
- $F_3^6 := \{e_t\partial e_{ijk} \mid e_{ijk} \in C_3', t \in [n] \setminus \{i,j,k\}, i,j,k$ are in the same $\overline{K_3^3}\}$.

Lemma 9. *For a signed graphic arrangement associated to a signed graph G not containing a subgraph isomorphic to B_2, we have*

$$\mathrm{span}(F_3) = \bigoplus_{i=1}^{6} \mathrm{span}(F_3^i).$$

Lemma 10. $\dim(\mathrm{span}(F_3^2)) = 10k_4$, $\dim(\mathrm{span}(F_3^3)) = 10d_3$, $\dim(\mathrm{span}(F_3^4)) = 10g_\circ$, $\dim(\mathrm{span}(F_3^5)) = 19d_{3,1}$ *and* $\dim(\mathrm{span}(F_3^6)) = 10k_{3,3}$.

Then we can get $\dim(I_2^3)$.

Lemma 11. *For a signed graphic arrangement associated to a signed graph G not containing a subgraph isomorphic to B_2, we have*

$$\dim(I_2^3) = (n-2)(k_3 + d_{2,1} + k_{3,3}) - 2k_4 - 2d_3 - 2g_\circ - 2k_{3,3} - 5d_{3,1}.$$

Proof of Main Theorem 6.

$$\phi_3 = 2\binom{n+1}{3} - n(\binom{n}{2} - k_3 - d_{2,1} - k_{2,2}) + \binom{n}{3} - \dim(I_2^3).$$

Because $2\binom{n+1}{3} - n\binom{n}{2} + \binom{n}{3} = 0$, then from Lemma 11 we obtain

$$\phi_3 = 2(k_3 + k_4 + d_3 + d_{2,1} + k_{2,2} + k_{3,3} + g_\circ) + 5d_{3,1}.$$

□

References

[1] Michael Falk. Arrangements and cohomology. *Annals of Combinatorics*, 1(1):135–157, 1997.

[2] Michael Falk. Combinatorial and algebraic structure in Orlik–Solomon algebras. *European Journal of Combinatorics*, 22(5):687–698, 2001.

[3] Henry Schenck and Alexander Suciu. Lower central series and free resolutions of hyperplane arrangements. *Transactions of the American Mathematical Society*, 354(9):3409–3433, 2002.