C*-algebras associated with self-similar group actions

Keisuke Yoshida
Hokkaido University

1. Abstract

A Banach $*$-algebra A with C*-norm (i.e. $\|a^*a\| = \|a\|^2$ for any $a \in A$) is called C*-algebra. It is well known that C*-algebras are deeply related to theories of topological dynamical systems.
Today I bring an example. In [1], V. Nekrashevych showed that every self-similar group actions on the Cantor set generates C*-algebras. I introduce Nekrashevych's construction and the properties of the algebras.

2. Self-similar group action

Notation 2.1. Let X be a finite set with $|X| = d$. Write $X := \{1, 2, \cdots, d\}$. X^* is the set of finite words over X. i.e. $X^* = \bigcup_{n\geq 0} X^n$ where X^0 is the empty word. X^ω denotes the set of unilateral infinite words and it is equipped with the topology of the direct Tychonoff product of discrete sets X. One can show that X^ω is homeomorphic to the Cantor set since X^ω is totally disconnected, compact and without isolated points.

Definition 2.2 (self-similar group action)**.** A faithful action of a group G on X^ω is said to be *self-similar* if for every $g \in G$ and $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$g(xw) = yh(w) \tag{1}$$

for any $w \in X^\omega$. An easy calculation shows that $h \in G$ and $y \in X$ are uniquely determined by $g \in G$ and $x \in X$.

Remark 2.3. Using equation (1) several times, we see that for every $g \in G$ and $u \in X^n$ there exist $h \in G$ and $v \in X^n$ such that

$$g(uw) = vh(w)$$

for any $w \in X^\omega$.
v and h are also uniquely determined. Hence we write $v = v(g, u)$ and $h = h(g, u)$.

Definition 2.4 (level-transitive group action)**.** A self-similar group action on X^ω is said to be *level-transitive* if for any $n \in \mathbb{N}$ and $u \in X^n$

$$v(\cdot, u)\colon G \to X^n$$

is surjective.

Example 2.5. Consider the case $X = \{0, 1\}$, $G = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. Take generators $\alpha, \beta \in G$ with $\alpha^2 = \beta^2 = e$. Let a and b are homeomorphisms on X^ω defined by the rules

$$a(0w) = 1w, b(0w) = 0a(w)$$
$$a(1w) = 0w, b(1w) = 1b(w)$$

where $w \in X^\omega$ is arbitrary. $\alpha \mapsto a$ and $\beta \mapsto b$ define a injective group homomorphism. Clearly the action is self-similar and an inductive argument shows that it is level-transitive.

3. Nekrashevych's construction

In this section and next section, suppose that action of G on X^ω is level-transitive. We also assume that G is countable. Let Φ_R be the free right module over the group algebra $\mathbb{C}G$ with the free basis X. Define an $\mathbb{C}G$-valued inner product on Φ_R by

$$\langle\sum_{x\in X} x \cdot a_x \,, \sum_{x\in X} x \cdot b_x\rangle := \sum_{x\in X} a_x^* b_x$$

where $a_x, b_x \in \mathbb{C}G$ for any $x \in X$.
We can construct left module structure from self-similarity. It is given by

$$g(\sum_{x\in X} a_x \cdot x) := \sum_{x\in X} y(g, x) \cdot (h(g, x)a_x)$$

where $g \in G$and $a_x \in \mathbb{C}G$for any $x \in X$. Extending the above definition to the linear span, we get $\mathbb{C}G$-bimodule. We use the symbol Φ for this bimodule instead of Φ_R. To get a C*-algebra, we consider completions of $\mathbb{C}G$ with respect to C*-norms.

Definition 3.1 (self-similar completion of $\mathbb{C}G$)**.** A completion A of $\mathbb{C}G$ with respect to some C*-norm is said to be *self-similar* if Φ extends to an A-bimodule.

We need a definition and a notation to construct an example of self-similar completion.

Definition 3.2 (G-generic point)**.** A point $w \in X^\omega$ is *G-generic* if for any $g \in G$ either $g(w) \neq w$ or there exists open neighborhood U of w consisting of the points fixed under the action of g.

It is not difficult to see that the set of all G-generic points is the intersection of a countable family of open dense sets since G is countable. In particular the set of all G-generic points is not empty.

Notation 3.3. $G(w)$ is the G-orbit of $w \in X^\omega$. Let $l^2(G(w))$ be the Hilbert space of the l^2-functions on $G(w)$. Write $\pi_w\colon \mathbb{C}G \to B(l^2(G(w)))$ for a representation of $\mathbb{C}G$ given by $g(\delta_{h(w)}) := \delta_{gh(w)}$ where $\{\delta_{h(w)}|h \in G\}$ is the canonical orthonornal basis of $l^2(G(w))$. Moreover $\|\cdot\|_w$ is the operator norm on $\mathbb{C}G$ defined by the representation π_w and by A_w we denote the completion of $\mathbb{C}G$ with respect to $\|\cdot\|_w$.

Nekrashevych has proved the following theorems in [1]. Their proofs are based on functional analysis arguments, and therefore we do not see the details today.

Theorem 3.4. Let $w_1, w_2 \in X^\omega$ and suppose that w_1 is G-generic. Then for any $a \in \mathbb{C}G$

$$\|a\|_{w_1} \leq \|a\|_{w_2}.$$

In particular, if w_2 is also G-generic then

$$A_{w_1} \simeq A_{w_2}.$$

Write $A_\Phi := A_w$ with a G-generic point $w \in X^\omega$. This notation does not depends on the choice of w.

Theorem 3.5. A_Φ is self-similar.

We have got an example of self-similar completion. We define the Nekrashevych's algebra in the following way.

Definition 3.6. $\mathcal{O}_\Phi$ is the universal C*-algebra generated by A_Φ and $\{S_x|x \in X\}$ satisfying the following relations:

$$S_x^* S_y = \delta_{x,y} \tag{2}$$

$$\sum_{x\in X} S_x S_x^* = 1 \tag{3}$$

$$aS_x = \sum_{y\in X} S_y \langle y, a \cdot x\rangle \tag{4}$$

where $\delta_{x,y}$ is the Kronecker delta and $a \in A_\Phi$, $x, y \in X$ are arbitrary.

4. Properties of $\mathcal{O}_\Phi$

Nekrashevych has also shown the following theorem in [1].

Theorem 4.1. For any non-zero $a \in \mathcal{O}_\Phi$ there exist $p, q \in \mathcal{O}_\Phi$ such that $paq = 1$.

The above theorem implies that $\mathcal{O}_\Phi$ has no proper ideal and therefore surjective $*$homomorphisms from the universality are always injective. Hence every C*-algebra generated by A_Φ and $\{S_x|x \in X\}$ satisfying (2), (3) and (4) is isomorphic to $\mathcal{O}_\Phi$.
Moreover the above theorem also implies that any unital corner of $\mathcal{O}_\Phi$ (i.e. nonzero C*-subalgebra of the form $p\mathcal{O}_\Phi p$ where p satisfies the condition $p^2 = p, p = p^*$) contains a copy of $\mathcal{O}_\Phi$. One may consider this as self-similarity in C*-algebra context.

Notation 4.2. The gauge action Γ of $\mathbb{R}$ on $\mathcal{O}_\Phi$ is given by

$$\Gamma_t(g) := g \text{ and } \Gamma_t(S_x) := e^{it} S_x$$

where $g \in G$, $t \in \mathbb{R}$ and $x \in X$.

We are interested in special functionals on C*-algebras called KMS state.

Definition 4.3. A state ϕ on $\mathcal{O}_\Phi$ is *β-KMS state* for $\beta > 0$ if

$$\phi(a\Gamma_{i\beta}(b)) = \phi(ba)$$

for any Γ-analytic element $a, b \in \mathcal{O}_\Phi$. By $K_\beta(\Gamma)$, we denote the set of all β-KMS states.

Applying a theory of Laca and Neshveyev in [2], we have the following fact.

Theorem 4.4. If $\beta \neq log|X|$ then $K_\beta(\Gamma)$ is empty. If there exist G-generic point $w \in X^\omega$ such that for any $g \in G$

$$g(w) = w \iff g(xw) = xw \text{ for any } x \in X.$$

Then $K_{log|X|}(\Gamma)$ is not empty.

For example, $K_{log2}(\Gamma)$ is not empty in the case of Example 2.5.

References

[1] Nekrashevych, Volodymyr V. *Cuntz-Pimsner algebras of group actions.* J. Operator Theory **52** (2004), no. 2, 223-249.

[2] Laca, Marcelo; Neshveyev, Sergey *KMS states of quasi-free dynamics on Pimsner algebras.* J. Funct. Anal. **211** (2004), no. 2, 457-482.