p-adic field

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1 $\mathbb{Q}_p$ as a completion of $\mathbb{Q}$

Here we will give a definition of a p-adic field $\mathbb{Q}_p$ as a completion of $\mathbb{Q}$, where p is a fixed prime. Fix a prime number p. Given $x \in \mathbb{Q}^\times$, put $x = p^m s/t$ with $m \in \mathbb{Z}$ and integers s, t prime to p. Then we define a function $\nu_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ as $\nu_p(x) = m$ for $x \in \mathbb{Q}^\times$, and also we put $\nu_p(0) = \infty$. This function ν_p has the following properties:

$$\nu_p(xy) = \nu_p(x) + \nu_p(y);$$

$$\nu_p(x+y) \geq \mathrm{Min}\{\nu_p(x), \nu_p(y)\};$$

$$\nu_p(-x) = \nu_p(x).$$

Here we have $\infty + x = x + \infty = \infty$ and $\infty \geq x$ for any $x \in \mathbb{Z} \cup \{\infty\}$.

Next define a function $\varphi_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ as $\varphi_p(x) = p^{-\nu_p(x)} = p^{-m}$ for $x \in \mathbb{Q}^\times$, and $\varphi_p(0) = 0$. This function φ_p has the following properties:

$$\varphi_p(xy) = \varphi_p(x)\varphi_p(y);$$

$$\varphi_p(x+y) \leq \varphi_p(x) + \varphi_p(y);$$

$$\varphi_p(-x) = \varphi_p(x).$$

Instead of the second property the narrower inequality $\varphi_p(x+y) \leq \mathrm{Max}\{\varphi_p(x), \varphi_p(y)\}$ is valid.

Now let us define a metric on $\mathbb{Q}$, which is different form the ordinary metric used for the construction of $\mathbb{R}$. Put $\rho(x, y) = \varphi_p(x - y)$. We can verify that $\mathbb{Q}$ is a metric space with respect to ρ. We call this metric ρ a ***p*-adic metric** on $\mathbb{Q}$ with a fixed prime p.

In general, recall that an infinite sequence $\{x_n\}_{n=1}^{\infty}$ in an arbitrary metric space (X, δ) is called a **Cauchy sequence** (with respect to δ) if for every $\epsilon > 0$ there exists a positive integer N such that $\delta(x_m - x_n) < \epsilon$ for every $m, n > N$. We say (X, δ) **complete** (with respect to δ) if every Cauchy sequence in X is convergent. As an example, $\mathbb{R}$ is complete with respect to the metric $|x - y|$.

We refer to the following theorem from general topology:

Theorem 1. (Completion of a metric space) *Given a metric space (X, δ), there exists a metric space (X^*, δ^*) with the following properties: (1) X^* is complete with respect to δ^*; (2)X is a subset of X^* and $\delta^* = \delta$ on X; (3)X is dense in X^*. Moreover, (X^*, δ^*) is unique for (X, δ) up to isomorphism.*

Let R be the set of all Cauchy sequences (with respect to ρ) in $\mathbb{Q}$, which forms a commutative ring, and I$\subset$ R be an ideal consisting of all the sequences convergent to 0. Then R/I is a ring. Furthermore, it will turn out to be a field. Let $\mathbb{Q}_p$ denote R/I. This is called the **p-adic field**. Cleary, it contains $\mathbb{Q}$ by means of identifying $\gamma \in \mathbb{Q}$ with $\{\gamma, \gamma, ...\}$. Now we can define a metric on $\mathbb{Q}_p$. Let $\eta = \{\eta_n\}$, $\xi = \{\xi_n\}$ be two elements of R. Then define $\rho^*(\eta, \xi) = \lim_{n\to\infty} \rho(\eta_n, \xi_n)$. This is a metric on $\mathbb{Q}_p$ satisfying $\rho^* = \rho$ on $\mathbb{Q}$. Then the completion of $(\mathbb{Q}, \rho)$ is nothing but $(\mathbb{Q}_p, \rho^*)$.

Example 1. Let $p = 17$, we have the following congruences:

$$4^2 + 1 \equiv 0(mod.17)$$

$$(4 + 2 \cdot 17)^2 + 1 = 38^2 + 1 \equiv 0(mod.17^2).$$

We can find by induction a sequence $\{\gamma_n\}_{n=1}^{\infty}$ such that $\gamma_n^2 + 1 \in 17^n\mathbb{Z}$ and $\gamma_{n+1} - \gamma_n \in 17^n\mathbb{Z}$. Then the element $\gamma = \{\gamma_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $\gamma_n^2 + 1 \to 0$, $\gamma = \lim_{n\to\infty} \gamma_n$ satisfies the equation $\gamma^2 + 1 = 0$. Thus $\mathbb{Q}_{17}$ contains an element which is a root of the equation $X^2 + 1 = 0$.

Bibliography. B. L. van der Waerden, Algebra II, Springer-Verlag(1967).