Embeddings of a complex torus into projective spaces

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1 Notations

$H := \{\tau \in \mathbb{C} \mid \mathrm{Im}\tau > 0\}$, $\mathbf{e}(x) := \exp(2\pi\sqrt{-1}x)$,
$\Omega(\tau) := \{n_1 + n_2\tau \mid n_1, n_2 \in \mathbb{Z}\} = (1, \tau)$, $E(\tau) := \mathbb{C}/\Omega(\tau)$.

2 Main Result

Theorem 1. The theta function $\theta(z,\tau)$ is defined by a series $\sum_{n\in\mathbb{Z}} \mathbf{e}(\frac{1}{2}n^2\tau + nz)$, which converges uniformly on any compact set in $\mathbb{C}\times H$. Hence, $\theta(z,\tau)$ is a holomorphic function on $\mathbb{C}\times H$.

Corollary 1. For any $m, n \in \mathbb{Z}$, $\theta(z + m\tau + n, \tau)$ is equal to $\mathbf{e}(-\frac{1}{2}m^2\tau - mz)\theta(z,\tau)$. If we fix τ, then $\theta(z) := \theta(z,\tau)$ belongs to the $\mathbb{C}$-vector space V_l:

$$\left\{ f(z) \,\middle|\, \begin{array}{l} f(z) \text{ are entire functions, } f(z + lm\tau + ln) = \\ \mathbf{e}(-\frac{1}{2}l^2m^2\tau - lmz)f(z) \quad (\forall m, n \in \mathbb{Z}) \end{array} \right\}.$$

We show that theta functions form a basis of V_l.

Proposition 1. The dimension of the $\mathbb{C}$-vector space V_l is l^2. Let $(a_i, b_j) \in (\frac{1}{l}\mathbb{Z}) \times (\frac{1}{l}\mathbb{Z})$ $(i, j = 0, 1, 2, ..., l-1)$ be a complete system of representatives of $((\frac{1}{l}\mathbb{Z})/\mathbb{Z}) \times ((\frac{1}{l}\mathbb{Z})/\mathbb{Z})$. Then $\theta_{a_i,b_j}(z) := \mathbf{e}(\frac{1}{2}a_i^2\tau + a_i(z + b_j))\theta(z + a_i\tau + b_j)$ form a basis of V_l.

Lemma 1. Any non zero element of V_l has just l^2 zeros on $\mathbb{C}/l\Omega(\tau)$ with multiplicity.

Lemma 1 implies that if any function $f(z) \in V_l$ is not identically zero, the quotient set $\{w \in \mathbb{C} \mid f(w) = 0\}/l\Omega(\tau)$ has l^2 elements.

Lemma 2. The function $\theta_{\frac{1}{2},\frac{1}{2}}(z)$ is an odd function and $\theta_{\frac{1}{2},\frac{1}{2}}(0) = 0$. The zero set of $\theta(z)$ is $\{(p + \frac{1}{2})\tau + (q + \frac{1}{2}) \mid p, q \in \mathbb{Z}\}$.

For simplicity, $\theta_i(z)$ stand for $\theta_{a_i,b_i}(z)$ for representatives $(a_i, b_i) \in (\frac{1}{l}\mathbb{Z}) \times (\frac{1}{l}\mathbb{Z})$ $(0 \le i \le l^2 - 1)$. From now on, let l be greater than or equal to 2.

Proposition 2. The zero set of $\theta_i(z)$ is

$$\{(-a_i + p + \frac{1}{2})\tau + (-b_i + q + \frac{1}{2}) \mid p, q \in \mathbb{Z}\}.$$

Especially, $\theta_i(z)$ and $\theta_j(z)$ $(i \ne j)$ have no common zero.

By Proposition 2, since $(\theta_0(lz), \theta_1(lz), ..., \theta_{l^2-1}(lz))$ is not the zero vector for every $z \in \mathbb{C}$, it gives a point $[\theta_0(lz), \theta_1(lz), ..., \theta_{l^2-1}(lz)]$ in the projective space $\mathbb{P}^{l^2-1}$. Then, we have an analytic map:

$$\Phi_l : \mathbb{C} \ni z \longmapsto [\theta_0(lz), \theta_1(lz), ..., \theta_{l^2-1}(lz)] \in \mathbb{P}^{l^2-1}.$$

By Corollary 1, Φ_l induces an analytic map:

$$\phi_l : E(\tau) \ni z \longmapsto [\theta_0(lz), \theta_1(lz), ..., \theta_{l^2-1}(lz)] \in \mathbb{P}^{l^2-1}.$$

Lemma 3. For any $a, b \in \frac{1}{l}\mathbb{Z}$, there exist $c_{ij} \in \mathbb{C}$ such that: $\phi_l\left(z + \frac{a\tau+b}{l}\right)$

$$= \left[\sum_{j=0}^{l^2-1} c_{0j}\theta_j(lz), ..., \sum_{j=0}^{l^2-1} c_{ij}\theta_j(lz), ..., \sum_{j=0}^{l^2-1} c_{(l^2-1)j}\theta_j(lz)\right].$$

Theorem 2. The analytic map ϕ_l is an embedding of the complex torus $E(\tau)$ into the projective space $\mathbb{P}^{l^2-1}$.

Proof. We have to show the following two properties:

(1) The analytic map ϕ_l is injective.

(2) For every $p \in E(\tau)$, the linear map $d_{\phi_l}(p) : T_p \longrightarrow T_{\phi_l(p)}$ is injective, where T_p is the tangent space of $E(\tau)$ at p and $T_{\phi_l(p)}$ is that of $\mathbb{P}^{l^2-1}$ at $\phi_l(p)$.

If (1) or (2) does not hold, then we can choose $l^2 + 1$ district points of $E(\tau)$ so that a non zero element of V_l vanishes at these points. This contradicts Lemma 1. □

3 Application

Theorem 3. (Chow's Theorem) An analytic submanifold of a projective space is algebraic.

By Theorem 3, $\phi_l(E(\tau))$ in $\mathbb{P}^{l^2-1}$ is an algebraic subvariety. Therefore, there exist homogeneous polynomials $f_i(x_0, x_1, ..., x_{l^2-1})$ $(1 \le i \le N)$ such that $\phi_l(E(\tau)) = V(f_1, f_2, ..., f_N) := \{[\mathbf{x}] \in \mathbb{P}^{l^2-1} \mid f_i(\mathbf{x}) = 0 \ (1 \le \forall i \le N)\}$.

For example, if l is equal to 2, $\phi_2(E(\tau))$ is a quartic curve $V(f_1, f_2)$ on the projective space $\mathbb{P}^3$, where

$$f_1 = f_1(x_1, x_2, x_3, x_4) = \theta_{0,0}(0)^2x_0^2 - \theta_{0,\frac{1}{2}}(0)^2x_1^2 - \theta_{\frac{1}{2},0}(0)^2x_2^2,$$
$$f_2 = f_2(x_1, x_2, x_3, x_4) = \theta_{0,0}(0)^2x_3^2 - \theta_{\frac{1}{2},0}(0)^2x_1^2 + \theta_{0,\frac{1}{2}}(0)^2x_2^2.$$

4 Bibliographies

[1] Umemura, H. , Elliptic function theory -Analysis of elliptic curve- (in Japanese). University of Tokyo Press, Tokyo, 2000.

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