Filtered module on Filtered ring

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1. Filetered ring and Filtered module

Definition 1 A family $F = \{F_pA \mid p \in \mathbf{N}\}$ of additive subgroups of a ring A is called a filtration of A if the following conditions are satisfied:

(1) $1 \in F_0A$.

(2) $F_pA \subset F_{p+1}A$.

(3) $(F_pA)(F_qA) \subset F_{p+q}A$.

(4) $A = \bigcup_{p \in \mathbf{N}} F_pA$.

Then we call (A, F) a **filtered ring**.

Let (A, F) be a filtered ring. We set

$$\operatorname{gr} A = \bigoplus_{p=0}^{\infty} F_pA/F_{p-1}A \qquad (F_{-1}A = 0)$$

$$\sigma_p \colon F_pA \to F_pA/F_{p-1}A \colon \text{canonical morphism}$$

An additive group $\operatorname{gr} A$ is endowed with a structure of a ring by

$$\sigma_p(a)\sigma_q(b) = \sigma_{p+q}(ab) \qquad (a \in F_pA, b \in F_qA)$$

Moreover a filtered ring (A, F) is called quasi commutative filtered ring if it is satisfting the following conditions:

(5) $[F_pA, F_qA] \subset F_{p+q-1}A$

then $\operatorname{gr} A$ is a commutative ring.

Definition 2 Let (A, F) be a filtered ring. Let M be an (left) A-module and asuume that a family $F = \{F_pN \mid p \in \mathbf{Z}\}$ of additive subgroups of M satisfying

(1) $F_pM = 0 \quad$ for $\quad p \ll 0$.

(2) $F_pM \subset F_{p+1}M$.

(3) $(F_pA)(F_qM) \subset F_{p+q}M$.

(4) $M = \bigcup_{p \in \mathbf{Z}} F_pM$.

Then we call (M, F) a **filtered A-module.**

Let (M, F) be a filtered A-module and we set

$$\operatorname{gr}^F M = \bigoplus_{p \in \mathbf{Z}}^{\infty} F_pM/F_{p-1}M$$

$$\tau_p \colon F_pM \to F_pM/F_{p-1}M \colon \text{canonical morphism}$$

An additive group $\operatorname{gr}^F M$ is endowed with a structure of a $\operatorname{gr} A$-module by

$$\sigma_p(a)\tau_q(b) = \tau_{p+q}(ab) \qquad (a \in F_pA, b \in F_qM)$$

Definition 3 (M, F) is called a goog filtered A-module , if $\operatorname{gr}^F M$ is a finitely generated $\operatorname{gr} A$-module .

2. Weyl algebra

We define Weyl alegebra. It is example of filtered ring. We suppose that K is field and char $K = 0$, $K[x] = K[x_1, \ldots, x_n]$.

We get a injective K-algebra morphism given by

$$m \colon K[x] \to \operatorname{End}_K K[x] \qquad ((m(f))(g) = fg)$$

Then we consider $K[x]$ is K-subalegabra of $\operatorname{End}_K K[x]$

$D \in \operatorname{End}_K K[x]$ is derivation if

$$D(fg) = D(f)g + fD(g) \qquad (\text{for any} f, g \in K[x])$$

and the set is denoted by $\operatorname{Der}_K K[x]$.

$\operatorname{Der}_K K[x]$is $K[x]$-module. For example, $\{\partial_i\}(1 \leqq i \leqq n)$ is given by

$$\partial_i \colon K[x] \to K[x] \qquad (\partial_i(f) = \frac{\partial f}{\partial x_i})$$

Then $\{\partial_i\}(1 \leqq i \leqq n)$ are derivations.

Proposition 4 $\operatorname{Der}_K K[x]$ is a free $K[x]$-module. Moreever $\{\partial_i\}_{1 \leqq i \leqq n}$ is basis of $\operatorname{Der}_K K[x]$.

We get Weyl algebra by Proposition4.

Definition 5 Weyl algebra is generated by $K[x]$ and $\operatorname{Der}_K K[x]$ as K-subalgebra of $\operatorname{End}_K K[x]$ and denoted by $A_n(K)$.

By Proposition4, $A_n(K)$is generated by $\{x_i\}$, $\{\partial_i\}(1 \leqq i \leqq n)$ and $A_n(K)$ has the following fundamental relations :

$$x_ix_j = x_jx_i,\ \partial_i\partial_j = \partial_j\partial_i,\ \partial_ix_j = x_j\partial_i + \delta_{ij}. \tag{1}$$

elements x^α and ∂^α of $A_n(K)$ are defined by

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},\ \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \quad (\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbf{N}^n)$$

Proposition 6 $\{x^\alpha\partial^\beta \mid \alpha, \beta \in \mathbf{N}^n\}$ is basis of $A_n(K)$ on K.

Let P be

$$P = \sum_{\alpha \in \mathbf{N}^n} f_\alpha \partial^\alpha \in A_n(K).$$

Then $\max\{|\alpha| \mid f_\alpha \neq 0\}$ is called **rank** of differential operator P and denoted by rank P. We define $A_n(K)$-subspace as $F_p = \{P \mid \text{rank}P \leq p\}$. If $p < 0$, then $F_p = 0$.

Proposition 7 $F = \{F_p \mid p \geqq 0\}$ is satisfying the following conditions:

(1) $1 \in F_0$.

(2) $F_p \subset F_{p+1}$.

(3) $F_pF_q \subset F_{p+q}$.

(4) $[F_p, F_q] \subset F_{p+q-1}$.

(5) $A_n(K) = \bigcup_{p \in \mathbf{Z}} F_p$.

A pair $(A_n(K), F)$ is a quasi commutative filtered ring by Proposition7. We set

$$\operatorname{gr} A_n(K) = \bigoplus_{p=0}^{\infty} F_p/F_{p-1}$$

$$\sigma_p \colon F_p \to F_p/F_{p-1} \subset \operatorname{gr} A_n(K) \colon \text{canonical morphism.}$$

By proposition7(3), the product on $\operatorname{gr} A_n(K)$ is given by

$$\sigma_p(a)\sigma_q(b) = \sigma_{p+q}(ab) \qquad (a \in F_p,\ b \in F_q)$$

Then $\operatorname{gr} A_n(K)$ is K-algebra.

Proposition 8 Let $K[y, \xi]$ be $2n$ variable polynomial ring . the K-algebra isomorphism $K[y, \xi] \to \operatorname{gr} A_n(K)$ is given by $y_i \mapsto \sigma_0(x_i)$, $\xi_i \mapsto \sigma_1(\partial_i)$.

3. Singular Support and Involutive Theorem

Definition 9 Let (A, F) be a quasi commutatuve filtered ring and (M, F) be a good filtered A-module. We suppose that $\operatorname{gr}A$ is Noetherian. Then we set

$$\operatorname{SS}(M) = \operatorname{Supp}_0(\operatorname{gr}^F M)$$

$$J_M = \sqrt{\operatorname{Ann}_{\operatorname{gr} A} \operatorname{gr}^F M} = \bigcap_{\mathfrak{p} \in \operatorname{SS}(M)} \mathfrak{p}$$

$\operatorname{SS}(M)$ is called a **Singular Support** of M and J_M is called a **Characteristic ideal** of M. Those definition are independent of choosing of a good filtration F.

Definition 10 R : commutative ring. A product $\{,\}$ on R is satisfing the following conditions :

$$\begin{aligned}
&\{a + b, c\} = \{a, c\} + \{b, c\},\ \{a, b + c\} = \{a, b\} + \{a, c\},\\
&\{a, b\} + \{b, a\} = 0,\\
&\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0,\\
&\{a, bc\} = \{a, b\}c + b\{a, c\}.
\end{aligned}$$

Then R is called a **Poisson ring**, $\{,\}$ is a **Poisson product**. A ideal I of R is **involutive** if it is satisfying that $\{I, I\} \subset I$

Example 11 (A, F): quasi commutative. A Poisson poduct on $\operatorname{gr} A$ is given by

$$\{\sigma_p(a), \sigma_q(b)\} = \sigma_{p+q-1}([a, b])$$

$$(\sigma_p \colon F_pA \to F_pA/F_{p-1}A \colon \text{canonical morphism})$$

Then $\operatorname{gr}A$ is a Poisson ring.

Example 12 Let K be characteristic 0 field and A be Weyl albebra $A_n(K)$, By Proposition 8, then a Poisson product on $\operatorname{gr} A$ is given by

$$\{f, g\} = \sum_{j=1}^{n} (\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}). f, g \in K[x, \xi]$$

Theorem 13 (Gabber's Involutive Theorem) A filtered ring(A, F) is satisfying the following the conditions:

1. A is $\mathbb{Q}$-algebra.
2. $\operatorname{gr} A$ is a commutative Noether ring.

and (M, F) is a good filtered A-module . Then any $\mathfrak{p} \in \operatorname{SS}(M)$ is involutive. Inparticular, J_M is involutive.

Theorem 14 Let (A, F) be filtered ring and $\operatorname{gr} A$ be regular ring. Suppose that M is an irreducible A-module and $\operatorname{gr} A$ has the following equation:

$$\dim(\operatorname{gr} A)_{\mathfrak{m}} = n \quad (\text{for any maximal ideal} \quad \mathfrak{m} \subset \operatorname{gr} A)$$

Then $\dim(\operatorname{gr} A)/\mathfrak{p}$ is independent of choosing of $\mathfrak{p} \in \operatorname{SS}(M)$

4. Geometric interpretation

We supose that $\operatorname{gr} A$ is $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$.

Definition 15 Let M be an A-module .

$$\operatorname{Ch}(M) = V(J_M) = \bigcup_{\mathfrak{p} \in \operatorname{SS}(M)} V(\mathfrak{p})$$

is called a characteristic variety. Then $\bigcup_{\mathfrak{p} \in \operatorname{SS}(M)} V(\mathfrak{p})$is called irreducible decomposition and $V(\mathfrak{p})$ is called an irreducible component. $V(I)$ is a corresponding variety of ideal I.

By Theorem14, if M is irreducible , any irreducibre component of $\operatorname{Ch}(M)$ has the same dimension. By Theorem13, we get the following equations :

$$\begin{aligned}
\operatorname{l.gl.dim}(A) &= \operatorname{r.gl.dim}(A)\\
&= n - \min\{\dim \operatorname{Ch}(M) | M \in \operatorname{Mod}^f(A), M \neq 0\}
\end{aligned}$$

Let A be $A_m(\mathbb{C})$. We get the following Theorem by Gabber's Involutive Theorem.

Theorem 16 Let M be a finitely generated $A_m(\mathbb{C})$-module. Then any irreducible component V of $\operatorname{Ch}(M)$ is satisfying that $\dim V \leq m$.

Definition 17 Let M be a finitely generated $A_m(\mathbb{C})$-module. Then M is called a holonomy module if $\dim \operatorname{Ch}(M) = m$.

Example 18 $A_m(\mathbb{C})$-module $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_m]$ is $\operatorname{gr} A_m(\mathbb{C})$-module by tivial filtration .The $\operatorname{Ch}(M)$ is given by

$$\operatorname{Ch}(\mathbb{C}[x]) = \{(x, 0) \mid x \in \mathbb{C}^m\}.$$

Thus $\mathbb{C}[x]$ is a holonomy module.

Theorem 19

$$\operatorname{l.gl.dim}(A_m(\mathbb{C})) = \operatorname{r.gl.dim}(A_m(\mathbb{C})) = m.$$

References

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