# The Universal Coefficient Theorem

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## 1. Definitions of Cohomology and Free Resolution

**Definition 1.1 (Cohomology)**
*Let $C = (C_n, \partial_n)_{n\in\mathbb{Z}}$ be a chain complex of free abelian groups, and $G$ be an abelian group. For each $n$, we define the following:*

$$C_n^* := \mathrm{Hom}(C_n, G),\ \ \partial_n^* : C_{n-1}^* \to C_n^*,\ f \mapsto f \circ \partial_n$$

*$C_n^*$ is called the $n$-cochain group of $C$ with coefficients in$G$, $\partial_n^*$ the $n$-coboundary map, $(C_n^*, \partial_n^*)_{n\in\mathbb{Z}}$ the cochain complex, and*

$$H^n(C;G) := \mathrm{Ker}\partial_{n+1}^*/\mathrm{Im}\partial_n^*$$

*the $n$-th cohomology group of $C$ with coefficients in $G$.*

**Example 1.1 (Singular Cohomology)**
*Let $X$ be a topological space, and $G$ be an abelian group. For each singular chain complex $S(X) = (S_n(X), \partial_n)_{n\in\mathbb{Z}_{\geq 0}}$,*

$$H^n(X;G) := H^n(S(X);G)$$

*is called the $n$-th singular cohomology of $X$ with coefficients in $G$.*

**Definition 1.2 (Free Resolution)**
*Let $H$ be an abelian group. Then,if the sequence*

$$F : \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

*satisfies $(1)F$is exact$(2)$each $F_n (n \geq 0)$is a free abelian group, $F$ is called a free resolution of $H$.*

**Example 1.2 (The example of free resolution)**
*For any free abelian group $H$, the following sequence is a free resolution of $H$:* $0 \to H \xrightarrow{id_H} H \to 0$

**Example 1.3 (The example of free resolution)**
*Let $n : \mathbb{Z} \to \mathbb{Z}$ be $n$times map, and $\pi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the canonical projetion, the following sequence is afree resolution of $\mathbb{Z}/n\mathbb{Z}$:*
$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$

## 2. Definition of $\mathrm{Ext}$ and its properties

To define $\mathrm{Ext}$,we introduce three lemmas.

**Lemma 2.1**
*For any abelian group $H$, the following free resolution of $H$.*

$$0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

**Lemma 2.2**
*Given free resolutions$F$ and $F'$ be a free resolution of abelian group $H$ and $H'$, then every homomorphism $\alpha : H \to H'$ can be extended to achain map from $F$to $F'$.*

$$\begin{array}{ccccccccccc}
F : \cdots & \to & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \to & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\
F' : \cdots & \to & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & H' & \to & 0
\end{array}$$

*Furthermore, any two such chain maps extending $\alpha$ are chain homotopic.*

Appying for lemma $2.2$ in the case $H' = H$ and $\alpha = id_H$, we get the following lemma. Note that when $F$, a free resolution of $H$, is regaded as a chain complex, we define $H^n(F;G) := \mathrm{Ker} f_{n+1}^*/\mathrm{Im} f_n^*$.

**Lemma 2.3**
*For any two free resolutions $F$ and $F'$ of $H$, there are canonical isomorphisms $H^n(F;G) \cong H^n(F';G)\,(n \geq 0)$.*

By the above lemmas, $\mathrm{Ext}$ can be defined.

**Definition 2.1 ($\mathrm{Ext}$)**
*Let $G$,$H$ be abelian groups, and $F$ be the following free resolutin of$H$.*

$$F : 0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

*Then we define $\mathrm{Ext}(H,G) := H^1(F;G)$.*

For any abelian group $H$ there exists a free resolution by Lemma $2.1$, and for any two free resolutions of $H$, $F$ and $F'$, $H^1(F;G) \cong H^1(F';G)$ by Lemma $2.3$.Therefore Definition $2.1$ is well-defined.

**Lemma 2.4 (Properties of $\mathrm{Ext}$)**
*Let $H$,$H'$ and $G$ be abelian groups.Then1,2,3 holds.*
*1.* $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H,G) \oplus \mathrm{Ext}(H',G)$
*2.* $\mathrm{Ext}(H,G) = 0$ *if $H$ is free.*
*3.* $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$

**Example 2.1 (The calculation of $\mathrm{Ext}$)**
*Let $H$ be a finitely generated abelian group. Then,the following isomorphism exists by the structure theorem of finitely generated abelian group, $H \cong \mathbb{Z}^r \oplus (\mathbb{Z}/e_1\mathbb{Z})^{\alpha_1} \oplus \cdots \oplus (\mathbb{Z}/e_m\mathbb{Z})^{\alpha_m}$ ,where $r$,$\alpha_1$,$\cdots$, $\alpha_m$ are integers more than $0$, $e_1$,$\cdots$, $e_m$ are integers more than $2$, $e_1 \mid e_2 \mid \cdots \mid e_m$. Let $T := (\mathbb{Z}/e_1\mathbb{Z})^{\alpha_m} \oplus \cdots \oplus (\mathbb{Z}/e_m\mathbb{Z})^{\alpha_m}$. Then $\mathrm{Ext}(H,\mathbb{Z}) \cong \mathrm{Ext}(T,\mathbb{Z})$ holds by $1$, $2$ in Lemma $2.4$. Therefore by $1$, $3$ in Lemma $2.4$, we get $\mathrm{Ext}(H,\mathbb{Z}) \cong T$.*

## 3. The Universal Coefficient theorem

**Theorem 3.1 (The universal coefficient theorem)**
*Let $C = (C_n, \partial_n)_{n\in\mathbb{Z}}$ be a chain complex of free abelian groups, and $G$ be an abelian group. Then, $H^n(C;G)$ are determined by split exact sequences:*

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G) \to 0$$

*Therefore, $H^n(C;G) \cong \mathrm{Hom}(H_n(C), G) \oplus \mathrm{Ext}(H_{n-1}(C), G)$ holds, where $H_n(C)$ is $n$-th homology group of $C$ with coefficients in $\mathbb{Z}$, and $h$ is the following group homomorphism:*

*For each $\overline{\varphi} \in H_n(C)$, $h(\overline{\varphi}) : H_n(C) \to G$, $\overline{x} \mapsto \varphi(x)$*

The short exact sequence in Theolem $3.1$ is natural. That is, for any $\alpha$, chain map from $C$ to $C'$, the following diagram is commute. Note that $\alpha_*$ is an induced homomorphism of $\alpha$, and $\mathrm{Ext}(\alpha_*)$ is a group homomorphism whichi assigns $\overline{f \circ \alpha} \in \mathrm{Ext}(H_{n-1}(C'), G)$ to $\overline{f} \in \mathrm{Ext}(H_{n-1}(C), G)$.

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Ext}(H_{n-1}(C), G) & \to & H^n(C;G) & \xrightarrow{h} & \mathrm{Hom}(H_n(C), G) & \to & 0 \\
 & & \uparrow{\scriptstyle \mathrm{Ext}(\alpha_*)} & & \uparrow{\scriptstyle (\alpha^*)_*} & & \uparrow{\scriptstyle (\alpha_*)^*} & & \\
0 & \to & \mathrm{Ext}(H_{n-1}(C'), G) & \to & H^n(C';G) & \xrightarrow{h} & \mathrm{Hom}(H_n(C'), G) & \to & 0
\end{array}$$

**Example 3.1 (Orientable closed surface $M$ of genus $g$)**

$$H_n(M) = \begin{cases} \mathbb{Z} & (n = 0, 2) \\ \mathbb{Z}^{2g} & (n = 1) \\ 0 & (n \geq 3) \end{cases}$$

*Since $H_n(M)$ is free, by Lemma $2.4$, $\mathrm{Ext}((H_{n-1}(C), G) = 0$. By Theorem 3.1, for any abelian group $G$ $H^n(M;G) \cong \mathrm{Hom}(H_n(M), G)$ holds.Therefore*

$$H^n(M;G) = \begin{cases} G & (n = 0, 2) \\ G^{2g} & (n = 1) \\ 0 & (n \geq 3) \end{cases}$$

**Example 3.2 (Finite cell complex)**
*Let $X$ be a finite cell complex. Then the chain complex $C(X)$ whichi is given by $X$ is composed of finitely generated free abelian groups, $H_n(C(X))$ is finitely generated. Let $T_n(C(X))$ be the torsion subgroup of $H_n(C(X))$. By Example $2.1$ $\mathrm{Ext}(H_{n-1}(C(X)), \mathbb{Z}) \cong T_{n-1}(C(X))$holds. since $\mathrm{Hom}(H_n(C(X)), \mathbb{Z}) \cong H_n(C(X))/T_n(C(X))$, by Theorem $3.1$,*

$$H^n(C(X);\mathbb{Z}) \cong (H_n(C(X))/T_n(C(X))) \oplus T_{n-1}(C(X))$$

## References

[1] Allen Hatcher, *AlgebraicTopology*, Cambridge University-Press(2002), 190-198