Simple weak twisted modules for the Heisenberg vertex operator algebra based on Whittaker modules

Haruna Imai
Graduate School of Science, Department of Mathematics, Hokkaido University

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Introduction

Classification of finite simple groups have been completed in 1970s. The classification theorem says that every finite simple group is isomorphic to cyclic group of order prime numbers, alternating group of degree $n \geq 5$, simple group of Lie type or 26 sporadic groups. One of 26 sporadic groups which have the largest order is Monster group. Between elliptic modular function and Monster group, there are interesting relation which is called the Moonshine conjecture. In 1992, Richard Borcherds proved the conjecture by constructing the moonshine module vertex operator algebra $V^\natural$. In this poster, we wll study a representation of $V^G = \{v \in V \mid gv = v \text{ for any } g \in G\}$, where V is a vertex operator algebra and G is a finite subgroup of $\mathrm{Aut}\, V$. Indeed, a V^G-module is used when we construct $V^\natural$.

The following is a conjecture (cf.[1,5,8]).

Conjecture 1. Under some conditions on V, every simple V^G-module is contained in some g-twisted V-module.

The conjecture is confirmed for several examples by investigating the Zhu-algebra associated with V^G. Because, Theorem 2.1.2 and Theorem 2.2.1 in [7] says that for a vertex operator algebra V, there is one to one correspondence between the set of all isomorphism classes of simple V-modules and that of simple modules for the Zhu algebra associated to V. For example, let $M(1)$ be a Heisenberg vertex operator algebra and θ a certain automorphism of order 2. Set $M(1)^+$ as $M(1)^+ = M(1)^{<\theta>}$. The conjecture for simple $M(1)^+$-modules is confirmed in [1] by using the above way.

The conjecture makes sense for weak modules too. But there is no way for weak modules as Theorem 2.1.2 and Theorem 2.2.1 in [7] so far and we have a few examples for weak modules. So, in this poster, we will show an example for weak $M(1)^+$ modules to verify the conjecture. To construct such example, we will use Whittaker modules for Virasoro algebra of Felinska, Jaskolski and Kosztolowicz in [2,6] and refer to [5]. The following is the Main theorem of this poster.

Main theorem

Let M be a nonzero weak $M(1)^+$-module generated by a Whittaker vector of Felinska, Jaskolski and Kosztolowicz for ω. Then, M is simple and the following θ-twisted $M(1)$ weak modules are complete set of representatives of equivalence classes of simple weak $M(1)^+$-modules with a Whittaker vector of Felinska, Jaskolski and Kosztolowicz for ω ;

$$M(1, \zeta_{\frac{3}{2}}, \zeta_{\frac{1}{2}}) \cong (M(1, -\zeta_{\frac{3}{2}}, \zeta_{\frac{1}{2}}) \text{ for } (\zeta_{\frac{3}{2}}, \zeta_{\frac{1}{2}}) \in \mathbb{C}^\times \times \mathbb{C}$$

In this poster, we have two sections except for Introduction. First section is construction of $M(1)$ and second one is construction of modules in the Main theorem.

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Construction of a Heisenberg vertex operator algebra

We will construct the Heisenberg vertex operator algebra in reference to Chapter 6 in [4]. Let H be a one dimensional $\mathbb{C}$-vector space equipped with a non-degenerate symmetric bilinear form $\langle$-,-$\rangle$. We take $h \in H$ such that $\langle h,h \rangle$=1. Set a Lie-algebra

$$\hat{H} = H \otimes \mathbb{C}\lfloor t, t^{-1} \rfloor \oplus \mathbb{C}K$$

with the Lie bracket relations

$$\lfloor \alpha \otimes t^m, \beta \otimes t^n \rfloor = m\langle \alpha, \beta \rangle \delta_{m+n,0} K, \quad \lfloor K, \hat{H} \rfloor = 0$$

for $\alpha,\beta \in H$ and $m,n \in \mathbb{Z}$. $\alpha(n)$ denotes $\alpha \otimes t^n \in \hat{H}$ for $\alpha \in H$ and $n \in \mathbb{Z}$. Set two Lie subalgebras of $\hat{H}$ as $\hat{H}_{\geq 0} = \oplus_{n \geq 0} H \otimes t^n$ and $\hat{H}_{<0} = \oplus_{n<0} H \otimes t^n$. Set $\mathbb{C}\mathbf{1}$ as a one dimentional $\hat{H}_{\geq 0}$-module uniquely determined by $h(i) \cdot \mathbf{1} = 0$ for $i \geq 0$ and $K \cdot \mathbf{1} = \mathbf{1}$. Set $M(1)$ as a $U(\hat{H})$-module induced by the $U(\hat{H}_{\geq 0})$-module $\mathbb{C}\mathbf{1}$ where $U(\mathfrak{g})$ is the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. Using Poincaré-Birkhoff-Witt theorem, we have

$$M(1) = U(\hat{H}) \otimes_{U(\hat{H}_{\geq 0})} \mathbb{C}\mathbf{1} = U(\hat{H}_{<0}) \otimes_{\mathbb{C}} \mathbb{C}\mathbf{1}$$

We will put a vertex operator algebra structure onto $M(1)$. Set a vertex operator map $Y(-,x) : M(1) \to \mathrm{End}[[x, x^{-1}]]$ as following;

$$Y(\mathbf{1}, x) = 1, \quad Y(h(-1)\mathbf{1}, x) = \sum h(n) x^{-n-1}$$

Using iterate formula, we can inductively expand the domain of vertex operator. Then, $(M(1), Y(-,x), \mathbf{1})$ is a vertex algebra. Furthermore, let ω be $\frac{h(-1)^2}{2}$. We define $L(n) = \omega_{n+1}$ for $n \in \mathbb{Z}$. Then, $\mathbb{C}L(n) \oplus \mathbb{C}\mathbf{1}$ is a representation of Virasoro algebra. Actually, $(M(1), Y(-,x), \mathbf{1}, \omega)$ has a vertex operator algebra structure.

Construction of simple θ-twisted weak $M(1)$-modules based on Whittaker modules

Let H be same as the previous section. Set a twisted affine Lie algebra

$$\hat{H}[-1] = (H \otimes t^{\frac{1}{2}} \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}K$$

with the Lie bracket relations

$$\lfloor \alpha \otimes t^m, \beta \otimes t^n \rfloor = m\langle \alpha, \beta \rangle \delta_{m+n,0} K, \lfloor K, \hat{H}[-1] \rfloor = 0$$

for $\alpha,\beta \in H$ and $m,n \in \mathbb{Z} + \frac{1}{2}$. $\alpha(n)$ denotes $\alpha \otimes t^n \in \hat{H}[-1]$ for $\alpha \in H$ and $n \in \mathbb{Z} + \frac{1}{2}$. Let I be a left ideal of $U(\hat{H}[-1])$ generated by $h(n)$, $h(\frac{3}{2}) \mp \zeta_{\frac{3}{2}}$, $h(-\frac{1}{2}) - \frac{1}{\pm \zeta_{\frac{3}{2}}}(\zeta_{\frac{1}{2}} - \frac{1}{2} h(\frac{1}{2})^2)$ and $K - 1 \in U(\hat{H}[-1])$ for $n \geq \frac{5}{2}$, $\zeta_{\frac{3}{2}} \in \mathbb{C}^\times$ and $\zeta_{\frac{1}{2}} \in \mathbb{C}$. Then, we have a $\mathbb{C}$-algebra $M(1, \zeta_{\frac{3}{2}}, \zeta_{\frac{1}{2}}) = U(\hat{H}[-1])/I \neq 0$.

We also define an automorphism θ of $M(1)$ of order 2 determined by

$$\theta(\alpha_1(n_1) \cdots \alpha_k(n_k)\mathbf{1}) = (-1)^k \alpha_1(n_1) \cdots \alpha_k(n_k)\mathbf{1}$$

Denote $M(1)^+$ as a θ-invariant subalgebra of $M(1)$.

We wil put a θ-twisted weak $M(1)$-module onto $M(1, \zeta_{\frac{3}{2}}, \zeta_{\frac{1}{2}})$. For $\alpha \in H$, we denote $\alpha(x) \in \mathrm{End}\, M(1, \zeta_{\frac{3}{2}}, \zeta_{\frac{1}{2}})[[x^{\frac{1}{2}}, x^{\frac{-1}{2}}]]$ as $\alpha(x) = \sum_{i \in \frac{1}{2} + \mathbb{Z}} \alpha(i) x^{-i-1}$. For $u = \alpha_1(-i_1) \cdots \alpha_k(-i_k)\mathbf{1}$, we define $Y_0(u, x)$ as ;

$$Y_0(u, x) =: \frac{1}{(i_1 - 1)!} \frac{d^{i_1 - 1}}{dx^{i_1 - 1}} \alpha_1(x) \cdots \frac{1}{(i_k - 1)!} \frac{d^{i_k - 1}}{dx^{i_k - 1}} \alpha_k(x) :$$

Colons are normal ordering procedure, which defines as follow ;

$$: \alpha_1(i_1) := \alpha_1(i_1)$$

$$: \alpha_1(i_1) \cdots \alpha_k(i_n) := \begin{cases} : \alpha_2(i_2) \cdots \alpha_n(i_n) : \alpha_1(i_n) & \text{if } i_1 \geq 0 \\ \alpha_1(i_1) : \alpha_2(i_2) \cdots \alpha_n(i_n) : & \text{if } i_1 < 0 \end{cases}$$

We define $c_{mn} \in \mathbb{Q}$ for $m, n \in \mathbb{Z}_{\geq 0}$ by ;

$$\sum_{m,n \in \mathbb{Z}_{\geq 0}} c_{mn} x^m y^n = -\log(\frac{(1+x)^{\frac{1}{2}} + (1+y)^{\frac{1}{2}}}{2})$$

and set ;

$$\triangle_x = \sum_{m,n \in \mathbb{Z}_{\geq 0}} c_{m,n} h(m) h(n) x^{-m-n}$$

Then, for $u \in M(1)$ we define a vertex operator $Y_{M(1,\zeta_{\frac{3}{2}},\zeta_{\frac{1}{2}})}(-, x) \in \mathrm{End}[[x^{\frac{1}{2}}, x^{\frac{-1}{2}}]]$ by ;

$$Y_{M(1,\zeta_{\frac{3}{2}},\zeta_{\frac{1}{2}})}(u, x) = Y_0(e^{\triangle_x} u, x)$$

By Chapter 9 in [3], $(M(1, \zeta_{\frac{3}{2}}, \zeta_{\frac{1}{2}}), Y_{M(1,\zeta_{\frac{3}{2}},\zeta_{\frac{1}{2}})})$ is a θ-twisted weak module. Furthermore, we have a following lemma by using same arguments as Lemma 2.1 and Lemma 2.3 in [5].

Lemma 1. $(M(1, \zeta_{\frac{3}{2}}, \zeta_{\frac{1}{2}}), Y_{M(1,\zeta_{\frac{3}{2}},\zeta_{\frac{1}{2}})})$ is a simple weak $M(1)^+$-module for any $(\zeta_{\frac{3}{2}}, \zeta_{\frac{1}{2}}) \in \mathbb{C}^\times \times \mathbb{C}$.

Definition 1. Let V be a vertex operator algebra and M a weak V-module. $0 \neq w \in M$ is called a *Whittaker vector of Felinska, Jaskolski and Kosztolowicz for* ω, if there is $(\lambda_3, \lambda_1) \in \mathbb{C}^\times \times \mathbb{C}$ which satisfies following conditions ;

$$L(i)w = \begin{cases} \lambda_1 w & \text{if } i = 1 \\ \lambda_3 w & \text{if } i = 3 \\ 0 & \text{if } i \geq 4 \end{cases}$$

Remark 2. $\mathbf{1} \in M(1, \zeta_{\frac{3}{2}}, \zeta_{\frac{1}{2}})$ is a Whittaker vector of Felinska, Jaskolski and Kosztolowicz for ω of type $(\frac{1}{2}\zeta_{\frac{3}{2}}^2, \zeta_{\frac{1}{2}})$. And we have the map ;

$$\mathbb{C}^\times \times \mathbb{C} \to \mathbb{C}^\times \times \mathbb{C}, (\zeta_{\frac{3}{2}}, \zeta_{\frac{1}{2}}) \to (\frac{1}{2}\zeta_{\frac{3}{2}}^2, \zeta_{\frac{1}{2}})$$

is surjective and the image of $(\zeta_{\frac{3}{2}}, \zeta_{\frac{1}{2}})$ under this map are equal if and only if $\zeta_{\frac{3}{2}} = \pm \zeta'_{\frac{3}{2}}$.

Using Lemma 3.3 and proof of Theorem 1.1 in [5] and above Lemma 1, we have the Main theorem.

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