RPDEs, Feynman–Kac and the lift of Brownian motion

"Singular Stochastic PDEs "
CRM Pisa, September 2014

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1. **Motivation: revisit some old problems**
   1.1 Nonlinear filtering/Optimal control/SPDEs
   1.2 Rough paths
   1.3 RPDEs via Feynman–Kac
2. **Rough and Stochastic Differential Equations (RSDEs)**
   2.1 Lifting a BM *and* a rough path
   2.2 Applications

1. Motivation

Nonlinear filtering

$$\begin{cases} dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t & \text{(signal)} \\ dY_t = h(Y_t)\,dt + d\widetilde{B}_t & \text{(observation)} \end{cases}$$

- **Goal:** best estimator for $\varphi(X_T(\omega))$ given $Y(\omega)|_{[0,T]}$

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- Precisely: for $\varphi \in C\left(\mathbb{R}^{d_{\text{SIG}}}, \mathbb{R}\right)$, calculate

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- Basic measure theory: $\exists\phi : C\left([0, T], \mathbb{R}^{d_{\text{OBS}}}\right) \to \mathbb{R}$ such that

$$\mathbb{E}\left[\varphi\left(X_T\right)|\sigma\left(Y_r : r \in [0, T]\right)\right] = \phi\left(Y|_{[0,T]}\right) \text{ a.s.}$$

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- **Clark** '70s: $\exists! \phi^{\text{robust}}$ that is continuous in sup-norm IFF B and $\tilde{B}$ uncorrelated

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- Random measure

$$
\pi_T(\varphi) := \mathbb{E}\left[\varphi(X_T) \,|\, \sigma(Y_r : r \in [0, T])\right] = \phi\left(Y|_{[0,T]}\right)
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with density $\tilde{\rho}(T, x)\, dx$,

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- Then

$$d\rho(t, x) = L\left(t, x, \rho, D\rho, D^2\rho\right)dt + \sum_i \Lambda_i(t, x, \rho, D\rho) \circ dY_t^i$$

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- Gives (constructive!) robust version ϕ^{robust} if

$$Y \mapsto \rho$$

is continuous.

Optimal control

$$dX_t = \mu(X_t;\alpha_t)\,dt + \sum_i \sigma_i(X_t;\alpha_t)\,dB_t^i + \sum_i \overline{\sigma}_i(X_t)\,d\overline{B}_t^i$$

- **Goal:** Find

$$\alpha = \text{argmin}_{\alpha\in\mathcal{A}}\,\mathbb{E}\left[g(X_T) + \int_0^T f(X_r,\alpha_r)\,dr \middle| \overline{B}\right]$$

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- *Formally:* (Lions–Souganidis, Caruana–Friz–O; Diehl–Friz–Gassiat) value function

$$v(t,x) := \inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[g\left(X_T^{t,x}\right) + \int_t^T f\left(X_r^{t,x},\alpha_r\right)dr \middle| \overline{B}\right]$$

solves SPDE

$$dv = \inf_{\alpha\in\mathcal{A}} L\left(D^2 v, Dv, t, x; \alpha\right)dt + \sum_i \Lambda_i(t,x,v)\circ d\overline{B}^i_{T-t}$$

- $\overline{B} \mapsto v$ should be countinuous & defined pathwise

SPDEs

- Previous examples: need to understanding pathwise(!) SPDEs, regular solution map
- **Goal:** Well-posedness of scalar-valued parabolic PDEs driven by non-smooth paths η

$$\begin{cases} du &= L\left(t,x,u,Du,D^2u\right)dt + \sum_i \Lambda_i\left(t,x,u,Du\right)d\eta_t^i \quad \text{on} \\ u\left(0,x\right) &= u_0\left(x\right) \text{ on } \mathbb{R}^d \end{cases}$$

This talk: L and Λ linear

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This talk: L and Λ linear

- **Toy example**: $\eta = \left(B^1, B^2\right)$

$$du = \langle V_1\left(x\right), Du\rangle \circ dB_t^1 + \langle V_2\left(x\right), Du\rangle \circ dB_t^2, \ u\left(0,x\right) = u_0\left(x\right)$$

solved by $(t,x) \mapsto u_0\left(\phi\left(t,x\right)\right)$ where ϕ is flow of $\left(V_1, V_2\right)$

$$d\phi = -V_1\left(\phi\right)\circ dB_t^1 - V_2\left(\phi\right)\circ dB_t^2, \phi\left(t,x\right) = x$$

hence $B \mapsto u$ can only be **continuous in rough path metric.**

Rough paths...notation

- let $\eta \in C^1\left([0, T], \mathbb{R}^d\right)$ and define

$$\boldsymbol{\eta}_{s;t} := \left(\boldsymbol{\eta}^1_{s,t}, \boldsymbol{\eta}^2_{s,t}\right) := \left(\int_s^t d\eta, \int_s^t \int_s^{r_2} d\eta_{r_1} \otimes d\eta_{r_2}\right) \in \mathbb{R}^d \oplus \left(\mathbb{R}^d\right)^{\otimes 2}$$

let $\alpha \in (1/3, 1/2]$ and define

$$\rho_\alpha\left(\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}\right) := \sup_{s \neq t} \frac{\left|\boldsymbol{\eta}^1_{s,t} - \overline{\boldsymbol{\eta}}^1_{s,t}\right|}{|t-s|^\alpha} + \frac{\left|\boldsymbol{\eta}^2_{s,t} - \overline{\boldsymbol{\eta}}^2_{s,t}\right|^{1/2}}{|t-s|^{2\alpha}}$$

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- Completion of smooth paths wrt ρ_α is *the **space of geometric rough paths*** $\mathcal{C}^\alpha_d$, $(=\mathcal{C}^\alpha)$
- $\exists! y \in C\left([0,T],\mathbb{R}^e\right)$ s.t. $dy = V(y_t)\, d\boldsymbol{\eta}_t$ for $\boldsymbol{\eta}\in\mathcal{C}^\alpha$ and

$$\left(\mathcal{C}^\alpha,\rho_\alpha\right) \ni \boldsymbol{\eta} \mapsto y \in \left(C\left([0,T],\mathbb{R}^e\right),|.|_\infty\right)$$

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  is continuous
- $\boldsymbol{B} = \left(\int \circ dB, \int \circ dB \otimes \circ dB\right) \in \mathcal{C}^\alpha$

SPDEs

- Need to understand SPDEs pathwise(!) and their regularity
- **Goal:** Prove well-posedness of scalar-valued PDEs driven by **rough paths** η

$$\begin{cases} du & = & L\left(t, x, u, Du, D^2 u\right) dt + \Lambda\left(t, x, u, Du\right) d\boldsymbol{\eta}_t \quad \text{on } (0, \infty) \times \mathbb{R}^d \\ u\left(0, x\right) & = & u_0\left(x\right) \text{ on } \mathbb{R}^d \end{cases}$$

and study regularity

$$\boldsymbol{\eta} \mapsto u$$

A simple idea

- Let $dX_t = dB_t$. Then $u(t,x) = \mathbb{E}\left[\phi\left(X_T^{t,x}\right)\right]$ is the solution of

$$-du(t,x) = \frac{1}{2}D^2 u(t,x)\,dt, \;\; u(T,x) = \phi(x)$$

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- Let $\boldsymbol{\eta} \in \mathcal{C}^{\alpha}$ and "give meaning" to
$$dX_t = \sigma(X_t)\,dB_t + V(X_t)\,d\boldsymbol{\eta}_t.$$
Then $u(t,x) = \mathbb{E}\left[\phi\left(X_T^{t,x}\right)\right]$ should "solve"
$$-du = \frac{1}{2}\operatorname{Tr}\left[\left(\sigma^T\cdot\sigma\right)D^2 u\right]dt + \langle V, Du\rangle\, d\boldsymbol{\eta}_t.$$

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Remark

Pardoux '79, Rozovksii '83, Flandoli–Brzezniak '90, Nualart et. al '12

2. Rough and Stochastic Differential Equations (RSDEs)

- **Goal:** given $\boldsymbol{\eta} = (\boldsymbol{\eta}^1, \boldsymbol{\eta}^2) \in \mathcal{C}^\alpha$, construct a stochastic process $X = X^{\boldsymbol{\eta}}$ such that

$$" dX_t = \sigma(X_t) \circ dB_t + \overline{\sigma}(X_t)\, d\boldsymbol{\eta}_t "$$

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- We want:
  - $\boldsymbol{\eta} \mapsto |X^{\boldsymbol{\eta}}|_\infty$ is "nice" in $L^q(\Omega, \mathbb{P})$
  - $\sigma = 0$ RDE, $\overline{\sigma} = 0$ Ito–Stratonovich,
  - $\boldsymbol{\eta} = \left(\int \circ d\overline{B}, \int \circ d\overline{B} \otimes \circ d\overline{B}\right)(\omega)$ is lift of indep. BM $\overline{B}$, recover Ito–Stratonovich.

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- **Natural approach:** rough path lift $\mathbf{\Lambda}$ of $\Lambda = (B, \boldsymbol{\eta}^1)$ and solve the standard RDE

$$dX_t = (\sigma, \overline{\sigma})(X_t)\, d\mathbf{\Lambda}_t.$$

Bad+Good news

- **Problem:** given two rough paths

$$\boldsymbol{b} = \left(\boldsymbol{b}^1, \boldsymbol{b}^2\right) \in \mathcal{C}^\alpha \text{ and } \boldsymbol{\eta} = \left(\boldsymbol{\eta}^1, \boldsymbol{\eta}^2\right) \in \mathcal{C}^\alpha$$

$\nexists \boldsymbol{\lambda} \in \mathcal{C}^\alpha$ such that

$$\boldsymbol{\lambda} = \left(\begin{pmatrix} \boldsymbol{b}^1 \\ \boldsymbol{\eta}^1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{b}^2 & \text{"} \int \boldsymbol{b}^1 \otimes d\boldsymbol{\eta}^1 \text{"} \\ \text{"} \int \boldsymbol{\eta}^1 \otimes d\boldsymbol{b}^1 \text{"} & \boldsymbol{\eta}^2 \end{pmatrix} \right)$$

since the cross diagonal entries are not well-defined.

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since the cross diagonal entries are not well-defined.

- **Our situation**: more structure!

$$\boldsymbol{b} = \boldsymbol{B}(\omega) \equiv \left(\boldsymbol{B}^1, \boldsymbol{B}^2\right)(\omega) \equiv \left(\int \circ dB, \int \circ dB \otimes \circ dB \right)(\omega)$$

- $\int_0^t \boldsymbol{\eta}^1 \otimes d\boldsymbol{B}^1$ makes sense as a stochastic integral (Ito=Strat.) **and** if $\boldsymbol{\eta}$ is an indep. BM

$$\int_0^t \boldsymbol{B}^{1;j} \circ d\boldsymbol{\eta}^{1;i} \quad \equiv \int_0^t \boldsymbol{B}^{1;j} d\boldsymbol{\eta}^{1;i} \quad \equiv \boldsymbol{B}^{1;i} \boldsymbol{\eta}^{1;j} |_0^t - \int_0^t \boldsymbol{\eta}^{1;j} d\boldsymbol{B}^{1;i}$$

-> Take rhs as definition...

Theorem (Diehl–O–Riedel)

Let B be a d_1-dim. BM and $\boldsymbol{\eta} \in \mathcal{C}^{\alpha}_{d_2}$. Then there exists a stochastic process $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^{\eta}$ such that

1. $\boldsymbol{\Lambda} = \left(\boldsymbol{\Lambda}^1_t, \boldsymbol{\Lambda}^2_t\right)_{t \geq 0} \in \mathcal{C}^{\alpha}_{d_1+d_2}$ *(almost sure) and*

$$\boldsymbol{\Lambda}^1 = \left(\begin{array}{c} B \\ \boldsymbol{\eta}^1 \end{array} \right) \text{ and } \boldsymbol{\Lambda}^{2;ij} = \begin{cases} \int B^i \circ dB^j & \text{for } i,j \in \{1,\dots,d_1\} \\ \boldsymbol{\eta}^{2;(i-d_1),(j-d_1)} & \text{for } i,j \in \{d_1+1,\dots, \end{cases}$$

2. $\boldsymbol{\eta} \mapsto \left|\boldsymbol{\Lambda}^{\eta}\right|_{\rho_{\alpha'}}$ *is continuous in* $L^q\left(\Omega, \mathbb{P}\right)$ *and* $\left|\boldsymbol{\Lambda}^{\eta}\right|_{\rho_{\alpha'}}$ *has Gaussian tails (uniformly in* $\boldsymbol{\eta}$*),*
3. *If* $\boldsymbol{\eta} = \left(\int \circ d\overline{B}, \int \circ d\overline{B} \circ d\overline{B}\right)(\omega)$ *is the Stratonovich lift of an indep. BM* $\overline{B}$*, then* $\boldsymbol{\Lambda}$ *is the Stratonovich lift of* $\left(B, \overline{B}\right)$*.*

Proof (sketch).

- Define

$$\boldsymbol{\Lambda}^1 = \begin{pmatrix} B \\ \eta^1 \end{pmatrix} \ \textit{and } \boldsymbol{\Lambda}^{2;ij} = \begin{cases} \int B^i \circ dB^j & \textit{for } i,j \in \{1,\ldots, \\ \eta^{2;(i-d_1),(j-d_1)} & \textit{for } i,j \in \{d_1+1 \\ \int \eta^{1;(i-d_1)} dB^j & \textit{for } i \in \{d_1+1, \\ B^1\eta^1|_0^t - \int_0^t \eta^1 d B^1 & \textit{else} \end{cases}$$

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- Check that $\boldsymbol{\Lambda}$ is a rough path

Proof (sketch).

- Define

$$\mathbf{\Lambda}^1 = \begin{pmatrix} B \\ \boldsymbol{\eta}^1 \end{pmatrix} \ \textit{\textbf{and}}\ \mathbf{\Lambda}^{2;ij} = \begin{cases} \int B^i \circ dB^j & \textit{\textbf{for}}\ i,j \in \{1,\ldots, \\ \boldsymbol{\eta}^{2;(i-d_1),(j-d_1)} & \textit{\textbf{for}}\ i,j \in \{d_1+1 \\ \int \boldsymbol{\eta}^{1;(i-d_1)} dB^j & \textit{\textbf{for}}\ i \in \{d_1+1, \\ \boldsymbol{B}^1 \boldsymbol{\eta}^1 |_0^t - \int_0^t \boldsymbol{\eta}^1 d\boldsymbol{B}^1 & \textit{\textbf{else}} \end{cases}$$

- Check that $\mathbf{\Lambda}$ is a rough path

- To show that $|\mathbf{\Lambda}|_{\alpha'}$ has Gaussian tails for all $\alpha' < \alpha$ use that

$$\begin{aligned} \left|\boldsymbol{\eta}^{1;i} \boldsymbol{B}^{1;j}\right| &\leq |\eta|_\alpha |B|_\alpha |t-s|^{2\alpha} \\ \int \boldsymbol{\eta}^{1;i} d\boldsymbol{B}^{1;j} &\overset{\mathcal{L}}{=} \sqrt{\int (\eta_{s,u})^2\, du}.\mathcal{N}(0,1) \end{aligned}$$

- Let $\overline{B}$ be another BM carried on some $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$. To verify that for $\boldsymbol{\eta} = \left(\int d\overline{B}, \int \circ d\overline{B} \circ d\overline{B}\right)$ this gives Stratotonovich lift of

$$\left(B, \overline{B}\right) \text{ carried on } \left(\Omega \times \overline{\Omega}, \mathcal{F} \times \overline{\mathcal{F}}, \mathbb{P} \times \overline{\mathbb{P}}\right)$$

gives **null-set trouble** since we don't know if

$$(\overline{\omega}, \omega) \mapsto \boldsymbol{\Lambda}^{\overline{\boldsymbol{B}}(\overline{\omega})}(\omega)$$

is jointly measurable.

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  gives **null-set trouble** since we don't know if

$$(\overline{\omega}, \omega) \mapsto \boldsymbol{\Lambda}^{\overline{\boldsymbol{B}}(\overline{\omega})}(\omega)$$

  is jointly measurable.

- However, one can show that for $\overline{\mathbb{P}}$-a.e. $\overline{\omega}$,

$$\mathbb{P}\left(\omega : \boldsymbol{S}(\omega, \overline{\omega}) = \boldsymbol{\Lambda}^{\overline{\boldsymbol{B}}(\overline{\omega})}(\omega)\right) = 1$$

  where $\boldsymbol{S}$ is the standard Stratonovich lift of $\left(B, \overline{B}\right)$.

Remark

- $\eta \mapsto \mathbf{\Lambda}^\eta$ is measure-theoretic **ugly** (different η gives different null-sets for which $\mathbf{\Lambda}^2 = \ldots$ hold)
- But after averaging out, e.g. $\eta \mapsto \mathbb{E}\left[\mathbf{\Lambda}^\eta\right]$, **everything is nice!**

Remark

- $\boldsymbol{\eta} \mapsto \boldsymbol{\Lambda}^{\eta}$ is measure-theoretic **ugly** (different $\boldsymbol{\eta}$ gives different null-sets for which $\boldsymbol{\Lambda}^2 = \ldots$ hold)
- But after averaging out, e.g. $\boldsymbol{\eta} \mapsto \mathbb{E}\left[\boldsymbol{\Lambda}^{\eta}\right]$, **everything is nice!**
- Works because we need
  - consistency with Stratonovich only for **independent** Brownian motions ("$[\eta, B] = 0$")
  - $\boldsymbol{\eta} \mapsto \boldsymbol{\Lambda}^{\eta}$ only under expectations
  - Lyons '91: two-dimensional Gaussian process, Brownian marginals but iterated integrals only exist on null set
- Same approach works for (non-geometric) Ito lift, controlled paths (Gubinelli),...

Proposition

Let $(\eta^n) \subset C^1\left([0,T], \mathbb{R}^{d_2}\right)$ *such that*

$$\left(\int d\eta^n, \int d\eta^n \otimes d\eta^n\right) \to^{\rho_\alpha} \boldsymbol{\eta} \text{ as } n \to \infty$$

and X^n *the SDE solution of*

$$dX_t^n = \sigma\left(X_t^n\right) \circ dB_t + \overline{\sigma}\left(X_t^n\right) d\eta_t^n, \;\; X_0^n = x_0 \in \mathbb{R}^e$$

with $\sigma, \overline{\sigma} \in Lip^\gamma$ *for* $\gamma > \frac{1}{\alpha}$. *Then* $(X^n)_n$ *converges ucp to* $X^{\boldsymbol{\eta}}$, *the solution of the RDE*

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We call $X^{\boldsymbol{\eta}}$ the **RSDE solution**, and also write

$$dX_t^{\boldsymbol{\eta}} = \sigma\left(X_t^{\boldsymbol{\eta}}\right) \circ dB + \overline{\sigma}\left(X^{\boldsymbol{\eta}}\right) d\boldsymbol{\eta}_t$$

Proposition

Moreover,

1. $X^{\boldsymbol{\eta}}$ *depends only* $\boldsymbol{\eta}$ *and* B *but not on the choice of approximating sequence* (η^n),
2. *the solution map*

$$\boldsymbol{\eta} \mapsto \left|\left|X^{\boldsymbol{\eta}}\right|_\infty\right|_{L^q(\Omega,\mathbb{P})}$$

   is locally Lipschitz

3. $X^{\boldsymbol{\eta}}$ *has Gaussian tails, locally uniform in* $\boldsymbol{\eta}$: $\forall r > 0\ \exists \delta > 0$ *s.t.*

$$\sup_{|\boldsymbol{\eta}|_\alpha \leq r} \mathbb{E}\left[\exp\left(\delta \left|X^{\boldsymbol{\eta}}\right|_\infty^2\right)\right] < \infty.$$

4. *If* $\boldsymbol{\eta}$ *is a BM independent of* B, $X^{\boldsymbol{\eta}}$ *is the Stratonovich solution.*

2. Rough and Stochastic Differential Equations (RSDEs)

b) Applications

RPDEs revisited

Let

$$L : \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{S}^m \to \mathbb{R}, \quad G : \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$$

and $(\eta^n) \subset C^1\left([0,T], \mathbb{R}^d\right)$ converges to $\boldsymbol{\eta} \in C^\alpha$. If $\forall n$ $\exists!$ viscosity solution $v^n \in \mathrm{BUC}$ of

$$dv^n = L\left(x, v^n, Dv^n, D^2 v^n\right) dt + G\left(x, v^n, Dv^n\right) d\eta^n_t, v^n(0,x) = v_0(x)$$

and (v^n) converges to $v \in \mathrm{BUC}$, uniformly on compacts, and v depends only on $\boldsymbol{\eta}$, then we say that v is the solution of the RPDE

$$\begin{cases} dv &= L\left(t, x, v, Dv, D^2 v\right) dt + G\left(t, x, v, Dv\right) d\boldsymbol{\eta}_t \\ v(0,x) &= v(x). \end{cases}$$

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Remark

Lions–Souganidis '00, Caruana–Friz–O '10, Friz–O '12, Gubinelli–Tindel–Torreccilla '14,...

Theorem (Diehl–O–Riedel '14)

Let $\boldsymbol{\eta} \in \mathcal{C}^{\alpha}$ *and*

$$\begin{aligned} L(x, r, p, M) &= \operatorname{Tr}\left[\sigma(x)\sigma^{T}(x)M\right] + a(x)\cdot p \\ G_i(x, p) &= V_i(x)\cdot p \end{aligned}$$

with $V_i, \sigma \in Lip^{\gamma}$ *and* $a \in Lip^{\zeta}$ *for* $\gamma > \frac{1}{\alpha}$, $\zeta > 1$. *Then for every bounded, uniformly continuous* $\phi : \mathbb{R}^d \to \mathbb{R}$

1. $\exists!$ *RPDE solution* $u \in \mathrm{BUC}$ *of*

$$\begin{cases} -du &= L\left(x, u, Du, D^2 u\right)dt + G(x, Du)\, d\boldsymbol{\eta}_t \\ u(T, x) &= \phi(x). \end{cases}$$

2. *u is given as*

$$u(t, x) = \mathbb{E}\left[\phi\left(S_T^{t,x}\right)\right]$$

where $S^{s,x}$ *denotes the RSDE solution of*

$$\begin{cases} dS_t^{s,x} &= (\bar{a}, \sigma, V)\left(S_t^{s,x}\right) d\boldsymbol{\Lambda}_t^{\eta} \\ S_s^{s,x} &= x \end{cases}$$

Proof (sketch).

- for every $n \geq 1$ we have the Feynman–Kac representation

$$v^n(t,x) = \mathbb{E}\left[\phi\left(S_T^{n;t,x}\right)\right]$$

where v^n is the unique BUC solution of
$-dv^n = L(\dots)\,dt + G(\dots)\,d\eta_t^n$ and S^n is the SDE solution of

$$\begin{cases} dS_t^{n;s,x} &= \bar{a}\left(S^{n;s,x}\right)dt + \sigma\left(S^{n;s,x}\right) \circ dB_t + V\left(S^{n;s,x}\right)d\eta_t^n \\ S_s^{s,x} &= x \end{cases}$$

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- From joint lift

$$v^n(t,x) \equiv \mathbb{E}\left[\phi\left(S_T^{n;t,x}\right)\right] \to \mathbb{E}\left[\phi\left(S_T^{t,x}\right)\right] =: v(t,x)$$

and local uniform conv. follows from Arzela–Ascoli (the local uniform continuity follows from properties of $\mathbf{\Lambda}$ and standard RDE estimates).

Nonlinear filtering revisited

$$\begin{cases} dX_t = \mu(X_t)\,dt + \sum_k Z(X_t)\,dB_t^k + \sum_j L_j(X_t)\,d\overline{B}_t^j & \text{(signal)} \\ dY_t = h(X_t)\,dt + dB_t & \text{(observation)} \end{cases}$$

with $B, \overline{B}$ independent BM.

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with $B, \overline{B}$ independent BM.

Corollary

Let $\gamma > \frac{1}{\alpha}$ *for some* $\alpha \in \left(\frac{1}{2}, \frac{1}{3}\right)$,

$$\varphi \in \mathrm{Lip}^1, \mu \in \mathrm{Lip}^{1+\epsilon}, h, Z, L \in \mathrm{Lip}^\gamma.$$

Then $\exists!$ *continuous map*

$$\phi^{robust} : (\mathcal{C}^\alpha, \rho_\alpha) \to (\mathbb{R}, |.|)$$

such that

$$\mathbb{E}\left[\varphi(X_T)\,|\sigma(Y_r : r \in [0,T])\right] = \phi^{robust}(\boldsymbol{Y})$$

where $\boldsymbol{Y} = \left(\int \circ dY, \int \circ dY \circ dY\right)$.

THANKS FOR YOUR TIME!