Renormalisation group in SPDEs

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The KPZ equation

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- constant C_1 is of order $1/\varepsilon$.

The KPZ equation

This equation is given in $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$\partial_t u = \Delta u + (\partial_x u)^2 + \xi - C_1 - C' - C''$$

- constant C_1 is of order $1/\varepsilon$.
- Hidden structure : $C_2 = C' + C''$ where C' and C'' are of order $\log \varepsilon$ and C_2 is of order 1.

Convergence of the $\Pi_x^{(\varepsilon)}$

We need to give a meaning to $\Pi_x \tau$ where the τ are the monomials appearing in the decomposition of the solution. Sometimes this quantity is ill-defined.

- We use a space-time regularization for the noise replacing ξ by ξ_ε. We have to prove the convergence of $\Pi_x^{(\varepsilon)}$.
- Regularization of the product by hand :

$$\tilde{\Pi}_x^{(\varepsilon)} \mathcal{I}_1(\Xi)\mathcal{I}_1(\Xi) = \tilde{\Pi}_x^{(\varepsilon)} \mathcal{I}_1(\Xi) \tilde{\Pi}_x^{(\varepsilon)} \mathcal{I}_1(\Xi) - C_1$$

$$\tilde{\Pi}_x^{(\varepsilon)} \mathcal{I}_1(\mathcal{I}_1(\Xi)^2)^2 = (\tilde{\Pi}_x^{(\varepsilon)} \mathcal{I}_1(\mathcal{I}_1(\Xi)^2))^2 - C'$$

$$\tilde{\Pi}_x^{(\varepsilon)} \mathcal{I}_1(\Xi)\mathcal{I}_1(\mathcal{I}_1(\Xi)\mathcal{I}_1(\mathcal{I}_1(\Xi)^2)) =$$

$$\tilde{\Pi}_x^{(\varepsilon)} \mathcal{I}_1(\Xi) \tilde{\Pi}_x^{(\varepsilon)} \mathcal{I}_1(\mathcal{I}_1(\Xi)\mathcal{I}_1(\mathcal{I}_1(\Xi)^2)) - C''$$

Convergence of the renormalised model

There are two main issues :

- $(\tilde{\Pi}, \tilde{\Gamma})$ is a model.
- Proving the convergence for each tree.

$$
\begin{tikzcd}
\Pi_x^{(\varepsilon)} \arrow[r, "R_\varepsilon"] \arrow[d, dash] & \tilde{\Pi}_x^{(\varepsilon)} \arrow[r] \arrow[d, dash] & \tilde{\Pi}_x \arrow[d, dash] \\
u_\varepsilon \arrow[r, "R_\varepsilon"] \arrow[dr, "\hat{R}_\varepsilon"] & \tilde{u}_\varepsilon \arrow[r] & \tilde{u} \\
 & \hat{u}_\varepsilon \arrow[r] & \hat{u} \arrow[u, "\hat{R}^{-1}"']
\end{tikzcd}
$$

A recursive formulation

For any $L \in \mathcal{L}(T)$, we define a renormalisation map by :

$$
\begin{aligned}
M\mathbb{1} = \mathbb{1}, \ MX = X, \ M\Xi = \Xi \\
M \prod_i \tau_i = \prod_i M\tau_i - ML \prod_i \tau_i \\
M\mathcal{I}_k(\tau) = \mathcal{I}_k(M\tau)
\end{aligned}
$$

where the τ_i are elementary trees having the following form : Ξ, X and $\mathcal{I}_n(\sigma_i)$.

Admissible maps

Definition

Given a regularity structure (A, T, G), a map $L \in \mathcal{L}(T)$ is admissible if

1. *For each $\tau \in T$, $\|L\tau\| < \|\tau\|$.*
2. *For each $\tau \in T$, $|L\tau| > |\tau|$.*
3. *The following commutating property is satisfied :*

$$LG = GL.$$

We denote by $\mathcal{L}_{ad}(T)$ the set of admissible maps.

The map Π_x^M

We define Π_x^M by

$$(\Pi_x^M \Xi)(y) = (\Pi_x \Xi)(y), \quad (\Pi_x^M X)(y) = (\Pi_x X)(y).$$

Then we recursively define Π_x^M by

$$\Pi_x^M \mathcal{I}_k(\tau)(y) = (G^{(k)} * \Pi_x^M \tau)(y) - \sum_{i=0}^{|\mathcal{I}_k(\tau)|} \frac{(y-x)^i}{i!} (G^{(k+i)} * \Pi_x^M \tau)(x)$$

$$(\Pi_x^M \prod_i \tau_i)(y) = \prod_i (\Pi_x^M \tau_i)(y) - (\Pi_x^M (L \prod_i \tau_i))(y)$$

Proposition

The map Π^M satisfies : $(\Pi_x^M \tau)(x) = (\Pi_x M \tau)(x)$.

The renormalised model

The Γ^M operator is given by :

$$\Gamma^M_{xy}\mathcal{I}_k(\tau) = \mathcal{I}_k(\Gamma^M_{xy}\tau) - \sum_{j<|\tau|+2-k} (\Pi^M_x \mathcal{I}_{k+j}(\Gamma^M_{xy}\tau))(y)\frac{(X+x-y)^j}{j!}$$

Proposition

(Π^M_x, Γ^M) *is a model.*

Coproduct on trees

The coproduct on trees $\hat{\Delta} : T \to T \otimes T$ by :

$$\hat{\Delta}\mathbb{1} = \mathbb{1} \otimes \mathbb{1}, \qquad \hat{\Delta}X = X \otimes \mathbb{1} + \mathbb{1} \otimes X,$$
$$\hat{\Delta}\Xi = \Xi \otimes \mathbb{1} + \mathbb{1} \otimes \Xi, \qquad \hat{\Delta}(\tau\bar{\tau}) = (\hat{\Delta}\tau)(\hat{\Delta}\bar{\tau})$$
$$\hat{\Delta}(\mathcal{I}_k\tau) = (\mathcal{I}_k \otimes I)\,\Delta\tau + \mathbb{1} \otimes \mathcal{I}_k\tau.$$

For $\tau \in T$, we have $\hat{\Delta}\tau = \sum_i \tau_i^{(1)} \otimes \tau_i^{(2)}$. The $\tau_i^{(1)}$ are the patterns and the $\tau_i^{(2)}$ are the branches detached from the subtrees.

Explicit admissible maps

We define our general rules L by :

$$L\tau = (\ell \otimes I)\hat{\Delta}\tau$$

where the linear map $\ell : T \to \mathbb{R}$ recognizes some subtrees. We suppose that the support of ℓ is included in T^- in order to have for every $\tau \in T$, $|L\tau| > |\tau|$. Moreover $\ell(\tau)$ is zero if τ contains at least one X or if $\tau = \mathbb{1}$.

Example : the KPZ equation

We are able to describe the admissible maps L_{kpz} associated to the KPZ equation :

$$L_{kpz} = (\ell_{kpz} \otimes I)\hat{\Delta}$$

where ℓ_{kpz} is given by :

$$\ell_{kpz}(\mathcal{I}_1(\Xi)^2) = C_1, \qquad \ell_{kpz}\left(\mathcal{I}_1(\mathcal{I}_1(\Xi)^2)\right) = C'$$

$$\ell_{kpz}(\mathcal{I}_1(\Xi)\mathcal{I}_1(\mathcal{I}_1(\Xi)\mathcal{I}_1(\mathcal{I}_1(\Xi)^2))) = C''$$

Otherwise, ℓ_{kpz} is zero.

General Result

For any $L \in \mathcal{L}_{\hat{\Delta}}(T)$, we define a transformation M_L . We consider :

$$\mathcal{R}_{ad} = \langle M_L, L \in \mathcal{L}_{\hat{\Delta}}(T) \rangle$$

which is the group generated by the maps M_L.

Theorem

$\mathcal{R}_{ad}$ is a subgroup of $\mathcal{R}$ the renormalisation group.

The generalized KPZ equation

This equation is given in $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$\partial_t u = \Delta u + f(u)\,(\partial_x u)^2 + k(u)\,\partial_x u + h(u) + g(u)\,\xi.$$

- We obtain KPZ if $f \equiv g \equiv 1$ and $k \equiv h \equiv 0$.
- Contains the solution of the stochastic heat equation and invariant under composition with $C^\infty(\mathbb{R})$-functions
- We need to give a sense of the product of random distribution because of the lack of regularity in space.

Our guess

The renormalised equation for u is given by

$$\begin{aligned}
\partial_t u =& \partial_x^2 u + g(u)\left((\partial_x u)^2 - C_1 f(u)^2\right) + h(u) - C_2 g(u)^3 f(u)^4 \\
& f(u)\left(\xi - C_1 f'(u) - C_2 f'(u)^3 - C_3 f''(u) f'(u) f(u)\right) \\
& - (3C_2 + C_3)(g(u)^2 f'(u) f(u)^3 + g(u) f'(u)^2 f(u)^2) \\
& - C_3(g(u) f''(u) f(u)^3 + g'(u) f'(u) f(u)^3 + g'(u) g(u) f(u)^4) \\
& + k(u)\partial_x u
\end{aligned}$$

with C_2 and C_3 of order 1 and $C_1 \sim 1/\varepsilon$.

Admissible map for the generalized KPZ equation

The admissible map L_{kpzg} associated to the generalized KPZ is given by :

$$L_{kpzg} = (\ell_{kpzg} \otimes I)\hat{\Delta}$$

where ℓ_{kpzg} is defined by :

$$\ell_{kpzg}(\mathcal{I}_1(\Xi)^2) = C_1, \qquad \ell_{kpzg}(\mathcal{I}(\Xi)\Xi) = C_1$$

$$\forall \tau, \ \|\tau\| = 4 \wedge |\tau| < 0, \ \ell_{kpzg}(\tau) = C_\tau$$

Otherwise, ℓ_{kpzg} is zero.