

Convergence of the two-dimensional dynamic Ising-Kac model

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SINGULAR STOCHASTIC PDEs

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A famous result

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$X_\varepsilon \Rightarrow$ solution of the **KPZ equation** (w.r.t $\mathcal{D}(\mathbb{R}_+, \mathcal{C}(\mathbb{R}))$ topology).

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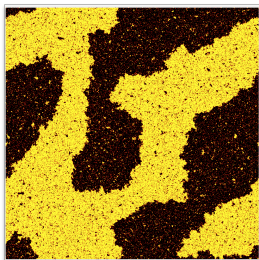
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- Scaling leaves stochastic heat equation invariant.
- Gives interpretation for **infinite renormalisation** in KPZ.
- Particle model changes with the scaling (asymmetry).

Kac-Ising model I



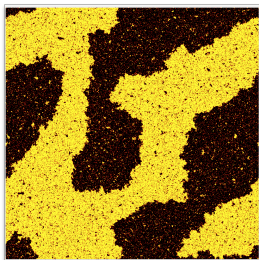
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$$\Lambda = \mathbb{Z}^d / (2N + 1)\mathbb{Z}^d \approx \{-N, \dots, N\}^d,$$

Spin configurations:

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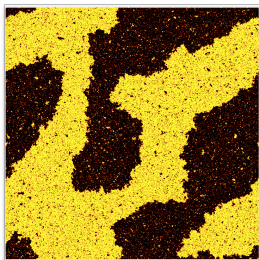
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Hamiltonian:

$$\mathcal{H}(\sigma) = -\frac{1}{2} \sum_{k, \ell} \kappa_\gamma(k - \ell) \sigma(k) \sigma(\ell),$$

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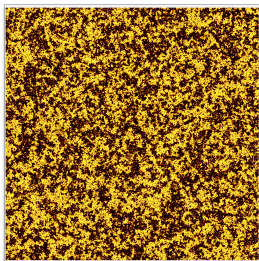
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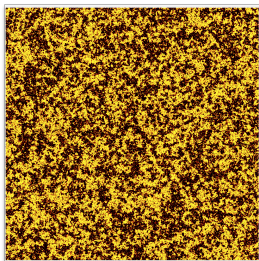
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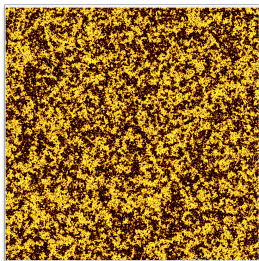
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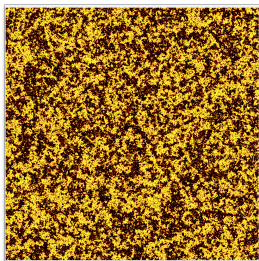
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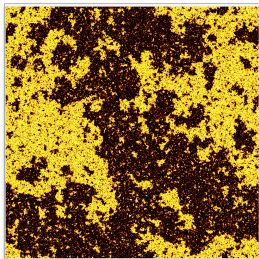


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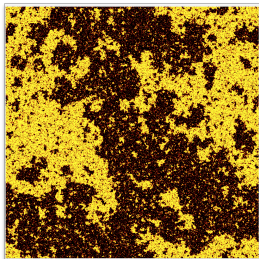
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Defines reversible Markov process.



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Aim: Describe non-linear fluctuations dynamical **near criticality** in dimension $d = 2$! Local averages described by SPDE

$$\partial_t X = \Delta X - X^3 + AX + \xi.$$

Local averages: $h_\gamma(\sigma(t), k) = \sum_{\ell \in \Lambda} \kappa_\gamma(\ell - k) \sigma(t, \ell)$.

Dynamic can be described in terms of h_γ :

$$\mathcal{H}_\gamma(\sigma) = -\frac{1}{2} \sum_{k \in \Lambda_N} \sigma(k) h_\gamma(\sigma, k),$$
$$c_\gamma(\sigma, j) = \frac{1}{2} \left(1 - \sigma(j) \underbrace{\tanh(\beta h_\gamma(\sigma, j))}_{\approx \beta h_\gamma(\sigma, j) - \frac{1}{3} \beta^3 h_\gamma(\sigma, j)^3} \right).$$

Evolution equation:

$$h_\gamma(t, k) = h_\gamma(0, k) + \int_0^t \mathcal{L}_\gamma h_\gamma(s, k) ds + m_\gamma(s, k),$$

where $m_\gamma(\cdot, k)$ martingale.

Coarse-grained field and rescaling II

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$$\begin{aligned} \mathcal{L}_\gamma h_\gamma(\sigma, \cdot) &= \left(\kappa_\gamma * h_\gamma(\sigma, \cdot) - h_\gamma(\sigma, \cdot) \right) + (\beta - 1) \kappa_\gamma * h_\gamma(\sigma, \cdot) \\ &\quad - \frac{\beta^3}{3} \left(\kappa_\gamma * h_\gamma(\sigma, \cdot) \right)^3 + \dots, \end{aligned}$$

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Predictable quadratic variation:

$$\langle m_\gamma(\cdot, k), m_\gamma(\cdot, j) \rangle_t = 4 \int_0^t \sum_{\ell \in \Lambda_N} \kappa_\gamma(k - \ell) \kappa_\gamma(j - \ell) \underbrace{c_\gamma(\sigma(s), \ell)}_{=\frac{1}{2} + \dots} ds.$$

Coarse-grained field and rescaling III

Rescaled dynamics: $X_\gamma(t, x) = \frac{1}{\delta} h_\gamma\left(\frac{t}{\alpha}, \frac{x}{\varepsilon}\right)$.

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Quadratic variation:

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For $d = 1$ result known:

Theorem (Bertini-Presutti-Rüdiger-Saada '93, Fritz-Rüdiger '95)

If $d = 1$ and

$$\varepsilon = \gamma^{\frac{4}{3}} \quad \delta = \gamma^{\frac{1}{3}} \quad \alpha = \gamma^{\frac{2}{3}} \quad (\beta - 1) = C\alpha,$$

then X_γ converges in law to the solution of the stochastic PDE

$$\partial_t X = \Delta X - \frac{1}{3} X^3 + CX + \sqrt{2}\xi,$$

where $\xi =$ space-time white noise.

Main result

Theorem (Mourrat, W. 2014+)

If $d = 2$ and

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then X_γ converges in law (w.r.t. $\mathcal{D}(\mathbb{R}_+, B_{\infty, \infty}^{-\nu-\kappa})$ topology) to the solution of the stochastic PDE

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- **Coarse graining** h_γ defined on the **same scale as interaction**. A posteriori

$$\mathbf{X}_\gamma(t, \varphi) = \sum_{x \in \Lambda_\varepsilon} \varepsilon^2 \varphi(x) \delta^{-1} \sigma(t/\alpha, x/\varepsilon),$$

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Higher dimensions:

- In principle scaling works also for $d = 3$. Hairer / Catellier-Chouk have stable notion of solutions for SPDE.
- For $d = 4$ scaling and SPDE theory break down.

Literature:

- Conjectured by Giacomin-Lebowitz-Presutti '99.
- In static case: trivial limit without renormalisation
Cassandro-Marra-Presutti '95, '97.
- Glimm-Jaffe-Spencer: Phase transition for static ϕ_2^4 '75.
Mass parameter A plays role of inverse temperature.
- Construction of da Prato-Debussche '03 crucial for our argument.

Renormalisation of Φ_2^4

$$\partial_t X_\delta = \Delta X_\delta - (X_\delta^3 - C_\delta X_\delta) + \xi_\delta.$$

$\xi_\delta(t, x) := \sum_{|\mathbf{k}| \leq \frac{1}{\delta}} e^{i\mathbf{k} \cdot \mathbf{x}} \dot{\beta}(t, \mathbf{k})$. Noise white in time, spatial correlations $\sim \delta$.

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In $d = 3$ recently by Hairer. Then $C_\delta = \frac{C_1}{\delta} + C_2 \log(\delta^{-1})$.

Stochastic heat equation:

$$\partial_t Z_\delta = \Delta Z_\delta + \sqrt{2} \xi_\delta \quad \text{on } [0, \infty) \times \mathbb{T}^d.$$

Solution $Z_\delta(t, x) = \int_0^t \int_{\mathbb{T}^d} K_\delta(t-s, x-y) \xi(ds, dy)$.

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$\mathcal{S}_x^\lambda \eta =$ scaled testfunction

$$\mathbb{E} \langle Z_\delta(t, \cdot), \mathcal{S}_x^\lambda \eta \rangle^2 = \int_0^t \int_{\mathbb{T}^d} \left(\int_0^t \int_{\mathbb{T}^d} K_\delta(t-s, x-y) \mathcal{S}_x^\lambda \eta(x) \right)^2 ds dy$$
$$\lesssim_{\text{unif. in } \delta} \begin{cases} \log(\lambda^{-1}) & \text{if } d = 2, \\ \lambda^{-d-2} & \text{if } d \geq 3. \end{cases}$$

Regularity of the linear part

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Gaussian \Rightarrow equivalent bounds in $L^p \Rightarrow$ regularity estimates.

Powers of the solutions

$$Z_\delta(t, x) = \int_0^t \int_{\mathbb{T}^d} K_\delta(t-s, x-y) \xi(ds, dy).$$

Itô formula:

$$\begin{aligned} Z_\delta(t, x)^3 &= \int_{([0, t] \times \mathbb{T})^3} \mathcal{W}_\delta^{(3)}(t, x; z_1, z_2, z_3) \xi(dz_1) \xi(dz_2) \xi(dz_3) \\ &\quad + 3Z_\delta(t, x) \underbrace{\int_{([0, t] \times \mathbb{T})} K_\delta(t-s, x-y)^2 ds dy}_{C_\delta}. \end{aligned}$$

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Nelson estimate \Rightarrow equivalent bounds in $L^p \Rightarrow$ regularity estimates.

Construction of the “model”:

For $d = 2$

Lemma

For every $p \geq 1, \nu > 0$

- $Z_\delta \rightarrow Z,$
- $Z_\delta^2 - C_\delta \rightarrow Z^{:2:},$
- $Z_\delta^3 - 3C_\delta Z_\delta \rightarrow Z^{:3:},$

in $L^p(C([0, T], \mathcal{B}_{\infty, \infty}^{-\nu}(\mathbb{T}^2)))$.

- $Z^{:n:}$ are called **Wick powers**.
- Key ingredient: **Equivalence of moments** in fixed Wiener chaos - Nelson estimate.
- Regularity measured in Besov spaces.

Evolution equation:

$$X_\gamma(t, x) \approx X_\gamma(0, x) + M_\gamma(t, x) + \int_0^t \left[\Delta_\gamma X_\gamma(s, x) - \frac{1}{3} K_\gamma *_\varepsilon \left(X_\gamma^3(s, x) - C_1 \log(\gamma^{-1}) X_\gamma(s, x) \right) \right] ds .$$

Linearised system:

$$R_{\gamma,t}(s, x) := \int_{r=0}^s P_{t-r}^\gamma dM_\gamma(r, x) \text{ where } P_t^\gamma = e^{t\Delta_\gamma} .$$

Approximate Wick powers:

$$R_{\gamma,t}^{:n:}(s, x) = n \int_{r=0}^s R_{\gamma,t}^{:n-1:}(r^-, x) dR_{\gamma,t}(r, x) .$$

Analysis of linearised particle system II

$$R_{\gamma,t}(s, x) := \int_{r=0}^s P_{t-r}^{\gamma} dM_{\gamma}(r, x)$$

$$R_{\gamma,t}^{:n:}(s, x) = n \int_{r=0}^s R_{\gamma,t}^{:n-1:}(r^-, x) dR_{\gamma,t}(r, x).$$

Lemma

For every n , the process $Z_{\gamma}^{:n:}(t) := R_{\gamma,t}^{:n:}(t, \cdot)$ converges in distribution to $Z^{:n:}$ with respect to $\mathcal{D}(\mathbb{R}_+, \mathcal{B}_{\infty, \infty}^{-\nu}(\mathbb{T}^2))$ topology.

- In tightness proof Nelson estimate replaced by martingale inequality (BDG).
- $\mathcal{D}(\dots)$ = space of càdlàg functions.

A Lemma for iterated integrals

$$\left(\mathbb{E} \sup_{0 \leq r \leq t} |(R_{\gamma, t}^{:n:}(r, x), f(x))|^p \right)^{\frac{2}{p}} \\ \lesssim \int_{r_1=0}^t \cdots \int_{r_n=0}^{r_{n-1}} \sum_{\mathbf{z} \in \Lambda_\varepsilon^n} \varepsilon^{2n} (\mathcal{W}^{(n)}(\cdot, \mathbf{r}, \mathbf{z}), f(\cdot))^2 d\mathbf{r} + \text{Err} ,$$

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and

$$\begin{aligned} \text{Err} = & C \varepsilon^4 \delta^{-2} \sum_{\ell=1}^n \int_{r_1=0}^t \cdots \int_{r_{\ell-1}=0}^{r_{\ell-2}} \sum_{z_1, \dots, z_{\ell-1} \in \Lambda_\varepsilon} \varepsilon^{2(\ell-1)} \\ & \left(\mathbb{E} \sup_{\substack{0 \leq r_\ell \leq r_{\ell-1} \\ z_\ell \in \Lambda_\varepsilon}} |(f(\cdot), R_{\gamma,t}^{:n-\ell:}(r_\ell, \cdot) \prod_{j=1}^{\ell} P_{t-r_j}^\gamma \star K_\gamma(\cdot - z_j))|^p \right)^{\frac{2}{p}} dr_{\ell-1}. \end{aligned}$$

2.) Non-linear evolution as **continuous** function of lifted Gaussian process

Regularisation trick: $v_\delta = X_\delta - Z_\delta$.

$$\begin{aligned}\partial_t v &= \Delta v_\delta - ((Z_\delta + v_\delta)^3 - 3C_\delta(Z_\delta + v_\delta)) \\ &= \Delta v_\delta - (Z_\delta^3 + 3Z_\delta^2 v_\delta + 3Z_\delta v_\delta^2 + v_\delta^3).\end{aligned}$$

Multiplicative inequality: If $\alpha < 0 < \beta$ with $\alpha + \beta > 0$

$$\|uv\|_{\mathcal{B}_{\infty,\infty}^\alpha} \lesssim \|u\|_{\mathcal{B}_{\infty,\infty}^\alpha} \|v\|_{\mathcal{B}_{\infty,\infty}^\beta}.$$

Used to deal with nonlinearity.

Can be repeated on level of X_γ .

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 - 1.) Treat a linearised version with probabilistic techniques.
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Conclusion

- Prove conjecture about two dimensional Kac-Ising model near criticality.
- The need for renormalisation for the SPDE is reflected in choice of scaling.
- Modern approach to irregular SPDE useful
 - 1.) Treat a linearised version with probabilistic techniques.
 - 2.) Treat the non-linear system as a perturbation.
- Core part of proof: Convergence of a linearised particle system **and nonlinear functions thereof.**