Multidimensional SDEs with distributional drift

Elena Issoglio

King's College London

September 2014 – Pisa, Singular Stochastic PDEs

Joint work with F. Flandoli and F. Russo

Heuristic and motivations
- SDE with distributional drift
- The virtual solution

Formalising the virtual solution
- A Kolmogorov-type PDE
- Regularity of the solution to the PDE
- The main result

Heuristic and motivations
SDE with distributional drift
The virtual solution

Formalising the virtual solution
A Kolmogorov-type PDE
Regularity of the solution to the PDE
The main result

The problem

We consider the multidimensional equation

$$dX_t = b(t, X_t)dt + dW_t, \quad X_0 = x_0$$

where:

- $(W_t)_t$ is a standard d-dimensional Brownian motion
- $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a vector field
- $(X_t)_t$ is the d-dimensional solution, i.e. it solves

$$X_t = x_0 + \int_0^t b(s, X_s)ds + W_t \qquad (\star)$$

We consider a class of drifts b which are *distributions* ($b \in \mathcal{S}'$)

Generalized drift: started in the '80 Harrison and Shepp, Portenko, Stroock and Yor, Le Gall, Engelbert and Schmidt...

$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$:

[Bass, Chen (2001)] one dimensional, time independent, σ and b Hölder continuous functions of order $> 1/2$

[Bass, Chen (2003)] multidimensional, time independent, $\sigma = 1$, b a measure of Kato class

[Flandoli, Russo, Wolf (2003 and 2004)] one dimensional, time independent, $\sigma > 0$ and b the derivative of a real continuous function.

[Flandoli, Gubinelli, Priola (2010)] multidimensional, time dependent, σ smooth bounded and b measurable unbounded function.

[Delarue, Diel (2014)] one dimensional, time dependent, $\sigma = 1$ and b derivative of α-Hölder continuous function, $\alpha > 1/3$

Main idea – heuristic

- Zvonkin: apply a transformation to the singular equation $dX_t = b(t, X_t)dt + dW_t$ to get an equation without drift.
- take the solution u to the Backward Kolmogorov equation $u_t + \frac{1}{2}\Delta u + b \cdot \nabla u = 0, \quad u(T) = x_0$
- Transformation: $\boxed{Y_t = u(t, X_t)}$ Formally (by Itô's formula)

$$
\begin{aligned}
dY_t &= u_t(t, X_t)dt + \nabla u(t, X_t)dX_t + \frac{1}{2}\Delta u(t, X_t)dt \\
&= \nabla u(t, X_t)dW_t
\end{aligned}
$$

BUT:

- $u_t + \frac{1}{2}\Delta u + b \cdot \nabla u = 0$
- need inverse u^{-1}
- need $\nabla u(t, u^{-1}(t, y))$ to be continuous, uniformly bounded and uniformly non-degenerate

Another transformation

Consider $\boxed{Y_t = \phi(t, X_t)}$ where

$$\phi(t, x) = x + u(t, x)$$

and u is the solution to

$$u_t + \frac{1}{2}\Delta u + b \cdot \nabla u - (\lambda + 1)u = -b, \quad u(T) = 0$$

Again formally

$$\boxed{Y_t = \phi(t, X_t)} \text{ and } \boxed{\phi(t,x) = x + u(t,x)}$$

By Itô's formula we get

$$\begin{aligned} dY_t =& u_t\, dt + (\nabla u + Id)\, dX_t + \frac{1}{2}\Delta u\, dt \\ =& (u_t + b \cdot \nabla u + b + \frac{1}{2}\Delta u)\, dt + (\nabla u + Id)\, dW_t \\ =& (\lambda + 1) u\, dt + \nabla u\, dW_t + dW_t \end{aligned}$$

with u evaluated at (t, X_t) that is $u = u(t, \phi^{-1}(t, Y_t))$

The virtual solution

To give a meaning to $\boxed{X_t = x + \int_0^t b(s, X_s)\, ds + W_t}$ $(\star)$

- consider the solution u to $u_t + \frac{1}{2}\Delta u + b \cdot \nabla u - (\lambda + 1)u = -b$
- apply Itô to $u(t, X_t)$
- define a *virtual solution* of $(\star)$ as the solution to

$$\begin{aligned} X_t = & x + u(0, x) - u(t, X_t) + (\lambda + 1) \int_0^t u(s, X_s)\, ds \\ & + \int_0^t \nabla u(s, X_s)\, dW_s + W_t \end{aligned} \qquad (\star\star)$$

- good notion of solution (consistent, independent of λ)

The plan

- Transform $(\star\star)$ through $Y_t = \phi(t, X_t)$
- $\exists!$ weak solution to $dY_t = (\lambda+1)u\,dt + \nabla u\,dW_t + dW_t$
- Use it to get the virtual solution $X_t = \phi^{-1}(t, Y_t)$

Problems

- solve the backward PDE with distributional coefficients
- show that the solution u is regular enough (function, differentiable in x, continuous, invertible, ...) to find a solution Y_t

Heuristic and motivations
SDE with distributional drift
The virtual solution

Formalising the virtual solution
A Kolmogorov-type PDE
Regularity of the solution to the PDE
The main result

The backward PDE

The PDE $\boxed{u_t + \frac{1}{2}\Delta u + b \cdot \nabla u - (\lambda+1)u = -b}$ with final condition $u(T) = 0$ is rewritten for $v(t) = u(T-t)$

$$v_t = \frac{1}{2}\Delta v + b \cdot \nabla v - (\lambda+1)v + b$$
$$v(0) = 0$$

The mild solution is given by

$$v(t) = \int_0^t P(t-r)(b(r) \cdot \nabla v(r))\, dr + \int_0^t P(t-r)(b(r) - \lambda v(r))\, dr$$

where $P(t)$ denotes the heat semigroup generated by $\frac{1}{2}\Delta - Id$

Problem

What is the meaning of the product $b(r) \cdot \nabla v(r)$*?*

Use the notion of paraproduct: for $f, g \in \mathcal{S}'$ we define the product

$$fg := \lim_{j \to \infty} S^j f \, S^j g$$

if the limit exists in $\mathcal{S}'$, where

- ρ is a mollifier with compact support
- $S^j f(x) := \left(\rho \left(\frac{\xi}{2^j} \right) \hat{f} \right)^{\vee} (x)$

Fractional Sobolev spaces on $\mathbb{R}^d$

- Let W_p^m denote the Sobolev space of order m on L_p, $m \in \mathbb{R}$. So for $f \in W_p^m$, its m-th weak derivative is in L_p
- Fractional Sobolev Spaces. Let $\alpha \in \mathbb{R}$

$$\boxed{H_p^\alpha(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : ((1+|\xi|^2)^{\alpha/2}\hat{f})^\vee \in L_p(\mathbb{R}^d) \right\}}$$

  endowed with the norm $\|f\|_{H_p^\alpha} = \|((1+|\xi|^2)^{\alpha/2}\hat{f})^\vee\|_{L_p}$
- for $\alpha < 0$: distributions
- for $\alpha \geq 0$: functions

The paraproduct

- $b \in H_q^{-\beta}$ distribution
- $\nabla v \in H_p^{\delta}$ function
- $0 < \beta < \delta$
- $q > p \vee \frac{d}{\delta}$ with $1 < p, q < \infty$

Then $b \cdot \nabla v \in H_p^{-\beta}$ and

$$\left\| b \cdot \nabla v \right\|_{H_p^{-\beta}} \leq c \left\| b \right\|_{H_q^{-\beta}} \left\| \nabla v \right\|_{H_p^{\delta}}$$

Remark: the product $b \cdot \nabla u$ is a distribution

The mild solution: a priori arguments

$$v(t) = \int_0^t P(t-r)\underbrace{\left(\overbrace{b(r)}\cdot\overbrace{\nabla v(r)}\right)}\,dr+\int_0^t P(t-r)\big(\underbrace{b(r)} - \underbrace{\lambda v(r)}\big)\,dr$$

The mild solution: a priori arguments

- look for a fixed point $v = I(v)$
-
-
-

$$v(t) = \int_0^t P(t-r)\underbrace{\left(\overbrace{b(r)}\cdot\overbrace{\nabla v(r)}\right)}\,dr + \int_0^t P(t-r)(\underbrace{b(r)} - \underbrace{\lambda v(r)})\,dr$$

-

The mild solution: a priori arguments

- look for a fixed point $v = I(v)$
- solution v is a weakly differentiable function
-
-

$$v(t) = \int_0^t P(t-r)\underbrace{\Big(\overbrace{b(r)}\cdot\overbrace{\nabla v(r)}\Big)}_{}^{\in H_p^\delta} dr + \int_0^t P(t-r)\Big(\underbrace{b(r)} - \underbrace{\lambda v(r)}_{\in H_p^{1+\delta}}\Big) dr$$

-

The mild solution: a priori arguments

- look for a fixed point $v = I(v)$
- solution v is a weakly differentiable function
- drift is a distribution
-

$$v(t) = \int_0^t P(t-r)\underbrace{\big(\overbrace{b(r)}^{\in H_q^{-\beta}} \cdot \overbrace{\nabla v(r)}^{\in H_p^{\delta}}\big)} \, dr + \int_0^t P(t-r)(\underbrace{b(r)}_{\in H_p^{-\beta}} - \underbrace{\lambda v(r)}_{\in H_p^{1+\delta}}) \, dr$$

-

The mild solution: a priori arguments

- look for a fixed point $v = I(v)$
- solution v is a weakly differentiable function
- drift is a distribution
- paraproduct is a distribution

$$v(t) = \int_0^t P(t-r)\underbrace{\big(\overbrace{b(r)}^{\in H_q^{-\beta}} \cdot \overbrace{\nabla v(r)}^{\in H_p^{\delta}}\big)}_{\in H_p^{-\beta}}\, dr + \int_0^t P(t-r)(\underbrace{b(r)}_{\in H_p^{-\beta}} - \underbrace{\lambda v(r)}_{\in H_p^{1+\delta}})\, dr$$

-

The mild solution: a priori arguments

- look for a fixed point $v = I(v)$
- solution v is a weakly differentiable function
- drift is a distribution
- paraproduct is a distribution

$$v(t) = \int_0^t P(t-r) \underbrace{\Big(\overbrace{b(r)}^{\in H_q^{-\beta}} \cdot \overbrace{\nabla v(r)}^{\in H_p^{\delta}} \Big)}_{\in H_p^{-\beta}} \, dr + \int_0^t P(t-r) \big(\underbrace{b(r)}_{\in H_p^{-\beta}} - \underbrace{\lambda v(r)}_{\in H_p^{1+\delta}} \big) \, dr$$

- the semigroup lifts almost 2 derivatives: from $-\beta$ to $1+\delta$

A fixed point argument

We solve $v = I(v)$ for

- $v(t) \in H_p^{1+\delta}$
- $b(t) \in H_{p,q}^{-\beta}$
- admissible drifts: $b \in L^\infty(0, T, H_{p,q}^{-\beta})$ for $0 < \beta < \delta$.
  (For example derivatives of Hölder continuous function of order $1/2+$)

Theorem

$\boxed{\text{There exists a unique } v \in C([0,T], H_p^{1+\delta})}$ with $q > p \vee \frac{d}{\delta}$

Moreover for $0 < \gamma < 1 - \delta - \beta$ the unique solution v is in

$$C^\gamma([0,T], H_p^{1+\delta})$$

Further regularity properties

Solve $v_t = \frac{1}{2}\Delta v + b \cdot \nabla v - (\lambda + 1)v + b$
and get mild solution $v \in C^\gamma([0, T], H_p^{1+\delta})$

- Sobolev embedding theorem:
  for $p > d/\delta$ then $v(t) \in C^{1,\alpha}$, $\alpha = \delta - d/p$
- for λ big enough we have $\|\nabla v\| \leq \frac{1}{2}$ uniformly
- the transformation $\phi(t, x) = x + u(t, x)$ is invertible
- ϕ^{-1} is jointly continuous

Putting everything together

Aim: solve $X_t = x + \int_0^t b(s, X_s)\, ds + W_t \qquad (\star)$

- Rewrite it as

$$X_t = x + u(0,x) - u(t,X_t) + (\lambda+1)\int_0^t u(s,X_s)\,ds + \int_0^t \nabla u(s,X_s)\,dW_s + W_t \qquad (\star\star)$$

- Transform it using $\phi(t,x) = x + u(t,x)$ to get

$$Y_t = y + (\lambda+1)\int_0^t u(s,\phi^{-1}(s,Y_s))\,ds + \int_0^t \nabla u(s,\phi^{-1}(s,Y_s))\,dW_s + W_t \qquad (\star\star\star)$$

Proposition

$\exists!$ weak solution Y to $dY_t = (\lambda+1)u\,dt + (\nabla u + \mathbf{1})\,dW_t$

- drift: continuous, linear growth
- diff coeff: continuous, uniformly bounded and non-degen

Main theorem

There exists a unique virtual solution X to

$$X_t = x + \int_0^t b(s, X_s)\,ds + W_t \qquad (\star)$$

with $b \in L^\infty(0, T;\ H^{-\beta}_{p,q})$ for $0 < \beta < 1/2$.

It is given by the weak solution $X_t = \phi^{-1}(t, Y_t)$ of $(\star\star)$

Summary

- take $b \in L^\infty(0, T; H_{p,q}^{-\beta})$ for $0 < \beta < 1/2$

$$dX_t = b(t, X_t)dt + dW_t$$

- virtual solution: rewrite it to give sense to the singular term
- transform it into

$$dY_t = (\lambda + 1)u\, dt + \nabla u\, dW_t + W_t$$

- $Y_t = \phi(t, X_t) = X_t + u(t, X_t)$ and u solution of

$$u_t + \frac{1}{2}\Delta u + b \cdot \nabla u - (\lambda + 1)u = -b$$

- solve for Y and transform back $X_t = \phi^{-1}(t, Y_t)$

- Bass, R. F. and Chen, Z.-Q., *Stochastic differential equations for Dirichlet processes*, Probab. Theory Related Fields (2001), 121(3), 422–446
- Bass, R. F. and Chen, Z.-Q. *Brownian motion with singular drift*, Ann. Probab. (2003), 31(2), 791–817
- Delarue, F. and Diel R. *Rough paths and 1d sde with a time dependent distributional drift. Application to polymers* (2014) arXiv:1402.3662
- Flandoli, F. Issoglio, E. Russo, F. *Multidimensional stochastic differential equation with distributional drift* (2014) arXiv:1401.6010
- Flandoli, F. Russo, F. and Wolf, J. *Some SDEs with distributional drift. II. Lyons-Zheng structure, Itô's formula and semimartingale characterization*, Random Oper. Stochastic Equations (2004), 12(2), 145–184
- Flandoli, F. Gubinelli, M. and Priola, E., *Well-posedness of the transport equation by stochastic perturbation*, Invent. Math. (2010), 180(1), 1–53
- Issoglio, E. *On a stochastic transport equation with fractal noise: existence, uniqueness and regularity of the solution*, Journal of Analysis and Its Applications, 32(1) (2013), 37-53.
- Runst,T. and Sickel, W. *Sobolev Spaces of Fractional Order, Nemytskij Operators, and nonlinear Partial Differential Equations*, de Gruyter Series in Nonlinear Analysis and Applications, Vol. 3, Walter de Gruyter & Co., Berlin, 1996.

Thank You!