Talk 22-1-2015 in Pisa

Highest weight vectors and transmutation

Preprint with the same title as my talk on ArXiv.

$k = \bar{k}$

$G = GL_n(k)$, T diagonal, U upper uni-triangular

$\mathfrak{g} = \mathfrak{gl}_n$: all $n\times n$ matrices $\mathcal{N} \subseteq \mathfrak{g}$ nilpotent cone

$G \curvearrowright \mathfrak{g}, \mathcal{N}$ by conjugation, so $G \curvearrowright k[\mathfrak{g}] = k[(x_{ij})_{ij}]$, $k[\mathcal{N}]$

$R \overset{\text{def}}{=} k[\mathfrak{g}]^G = k[s_1, \ldots, s_n]$ $s_i(A) = tr(\Lambda^i A)$

Problem Find finite homogeneous spanning sets (bases) for the R-modules $k[\mathfrak{g}]^U_\chi$ of highest weight vectors.

For $\chi \in \mathbb{Z}^n$ we have $k[\mathfrak{g}]^U_\chi \neq 0 \iff \chi$ is dominant (weakly decreasing) and has coord sum 0.

Such a χ can uniquely be written as $\chi = (\lambda_1, \lambda_2, \ldots, 0, \ldots, 0, \ldots, -\mu_2, -\mu_1) \overset{\text{def}}{=} [\lambda, \mu]$

where λ, μ are partitions of the same number with $l(\lambda) + l(\mu) \leq n$

Example $\partial_{1n} s_2, \ldots, \partial_{1n} s_n$ is an R-module basis of $k[\mathfrak{g}]^U_{[1,1]}$

($\partial_{1n} = \frac{\partial}{\partial x_{1n}}$)

We will proceed in three steps:

Step1 Reduction to the nilpotent cone

Step2 Transmutation

Step3 The transmuted problem.

Step1 Graded Nakayama Lemma: If M is a free graded R-module, then a subset of M the R-module M if its image in M/R^+M spans M/R^+M as a vector space.

In our case $k[\mathfrak{g}]/R^+k[\mathfrak{g}] = k[\mathfrak{g}]/(s_1, \ldots, s_n) = k[\mathcal{N}]$ and $k[\mathfrak{g}]^U_\chi / R^+ k[\mathfrak{g}]^U_\chi = k[\mathcal{N}]^U_\chi$.

New problem Find finite homogeneous spanning sets for the vector spaces $k[\mathcal{N}]^U_\chi$.

Step2 It will be convenient to work with the varieties $\mathcal{N}_m = \{A \in \mathcal{N} \mid A^{m+1} = 0\}$

Let r, s be integers with $r + s \leq n$. Consider the following map

$\varphi_m : Mat_n \longrightarrow Mat_{rs}^m$ given by

$$X \longmapsto \left(X_{r\rfloor, \lfloor s},\ (X^2)_{r\rfloor, \lfloor s},\ \dots,\ (X^m)_{r\rfloor, \lfloor s} \right)$$

For $S \in GL_n$ upper triangular we have

$$(SXS^{-1})_{r\rfloor, \lfloor s} = S_{r\rfloor, r\rfloor}\, X_{r\rfloor, \lfloor s} \left(S_{\lfloor s, \lfloor s}\right)^{-1}$$

and $\varphi_m(SXS^{-1}) = S_{r\rfloor, r\rfloor}\, \varphi_m(X) \left(S_{\lfloor s, \lfloor s}\right)^{-1}$

So φ_m induces a pullback map

$$k[Mat_{rs}^m]^{U_r \times U_s}_{(-\mu^{rev}, \lambda)} \longrightarrow k[\mathcal{N}_m]^U_{(\lambda, \mu)}$$ (I restricted φ_m to $\mathcal{N}_m$)

<u>Thm1</u> This map is surjective and for n sufficiently big an iso.

To prove this we need transmutation (<u>R. Brylinsky '93</u>)

Assume for convenience char $k = 0$.

Put $Z_{r,s,n} = \{(A,B) \in Mat_{rn} \times Mat_{ns} \mid AB = 0\}$

Then $k[Z_{r,s,n}] = \bigoplus_{\substack{\lambda,\mu\ |\lambda| = |\mu| \\ l(\mu) \le r,\ l(\lambda) \le s}} L_{GL_r}(-\mu^{rev}) \otimes L_{GL_s}(\lambda) \otimes L_{GL_n}([\lambda,\mu])^*$ (Veagne-Kashiwara)

So this defines a dictionary $L_{GL_r}(-\mu^{rev}) \otimes L_{GL_s}(\lambda) \longleftrightarrow L_{GL_n}([\lambda,\mu])$.

The variety $Z_{r,s,n}$ is the "catalyst" for the following transmutation process: $V \circlearrowleft GL_n \longmapsto Z_{r,s,n} \times^{GL_n} V \overset{def}{=} (Z_{r,s,n} \times V) /\!/ GL_n \circlearrowleft GL_r \times GL_s$

This simply replaces in $k[V]$ every copy of $L_{GL_n}([\lambda,\mu])$ by a copy of $L_{GL_r}(-\mu^{rev}) \otimes L_{GL_s}(\lambda)$.

The variety $Z_{r,s,n} \times^{GL_n} V$ is called the transmuted variety

Now put $W_m = \overline{\varphi_m(N_m)}$. Then $W_m \simeq Z_{r,s,n} \times^{GL_n} N_m$ (*)

It follows that φ_m must give an iso $k[W_m]^{U_r\times U_s}_{(-\mu^{rev},\lambda)} \xrightarrow{\sim} k[N_m]^{U}_{[\lambda,\mu]}$, because the dimensions are the same.
But restriction defines a surjection $k[Mat^m_{rs}]^{U_r\times U_s}_{(-\mu^{rev},\lambda)} \twoheadrightarrow k[W_m]^{U_r\times U_s}_{(-\mu^{rev},\lambda)}$
In char p this follows from the fact that (Mat^m_{rs}, W_m) is a good pair of $GL_r\times GL_s$-varieties. This proves Thm 1.

To prove (*) one shows that
$((A,B),X)\mapsto(AXB, AX^2B,\dots,AX^mB): Z_{r,s,n}\times N_m \to Mat^m_{rs}$
is GL_n-quotient morphism with image W_m
This requires some invariant theory (invariants of s vectors, r covectors and 1 matrix)

<u>Step 3</u> This leaves us with the task of describing the highest weight vectors in $k[Mat^m_{rs}]$. In char p I can only do this for special weights, but in char 0 it can be done in general.

<u>Put $V=k^r$, $W=k^s$</u>. Then $Mat_{rs} = V\otimes W^*$, $Mat^*_{rs} = V^*\otimes W$

$$k[Mat^m_{rs}] = \bigoplus_{t\geq 0} S^t((V^*\otimes W)^m) = \bigoplus_{t\geq 0,\ \nu\in\Sigma^t} S^\nu(V^*\otimes W)$$ where $S^\nu U = S^{\nu_1}U\otimes\cdots\otimes S^{\nu_m}U$.

($\nu\in\mathbb{N}_0^m$ with sum t)

$$= \bigoplus_{t\geq 0,\ \nu\in\Sigma_t} \left((V^*)^{\otimes t}\otimes W^{\otimes t}\right)_{Sym_\nu}$$

So $$k[Mat^m_{rs}]^{U_r\times U_s}_{(-\mu^{rev},\lambda)} \simeq \bigoplus_{t\geq 0,\ \nu\in\Sigma_t}\left(((V^*)^{\otimes t})^{U_r}_{-\mu^{rev}}\otimes (W^{\otimes t})^{U_s}_{\lambda}\right)_{Sym_\nu}.$$
where $t=|\lambda|=|\mu|$

By Schur-Weyl duality $((V^*)^{\otimes t})^{U_r}_{-\mu^{rev}}$ and $(W^{\otimes t})^{U_s}_\lambda$ are the Specht modules $S(\mu)$ and $S(\lambda)$

So we need to give bases for the coinvariants for a Young subgroup in the tensor product of two Specht modules.
This was done, with some mistakes, by J. Donin around 1990
Combinatorially this amounts to combining the well-known rule for restriction of Specht modules to Young subgroups with James-Peel['79] - Zelevinsky['81]'s notion of a "picture".
If we combine this with Thm 1 and the Graded Nakayama Lemma we obtain a finite homogeneous spanning set for the R-modules $k[\mathfrak{g}]^U_{[\lambda,\mu]}$. This result was already stated without proof by Donin. He actually claimed that these spanning sets are bases, but this is easily seen to be incorrect (using e.g. the charge on tableaux or the Hesselink-Peterson formula)
It is an interesting open problem (even combinatorial) to find subsets which are bases. Another open problem is to find bases (or "natural" spanning sets) for the vector spaces $k[\overline{O}]^U_\chi$, O a nilpotent orbit.
Donin also gave spanning sets for the spaces $(\Lambda\mathfrak{g})^U_{[\lambda,\mu]}$ as modules over $(\Lambda\mathfrak{g})^G$. The methods that I used don't immediately apply here.
If one applies "noncommutative transmutation" after somehow reducing $(\Lambda\mathfrak{g})^G$ to scalars, then the transmuted algebra should be the super symmetric algebra $SS(\mathrm{Mat}^m_{rs})$, where Mat^m_{rs} can be split in an odd and an even part in an obvious way.
In char p I can, at the moment, only give bases for the highest weight vectors in $k[\mathrm{Mat}^m_{rs}]$ for the weights $(-\mu^{rev},\lambda)$ when λ or μ is a row or when λ or μ is a column. In the first case the GL_m-module structure is that of an induced module, in the second case it is that of a Weyl module. This is in accordance with work of Akin-Buchsbaum-Weyman. Again we can combine this with Thm 1 to obtain spanning sets for the highest weight vectors of the corresponding weights in $k[N]$. In the first case they are actually bases. In the second case this is related, via Broer's Generalised Chevalley Restriction Theorem, to the coinvariant ring of the symmetric group Sym_n, studied for example by Garsia-Procesi.