Asymptotic Invariants of Links and Applications in MHD

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Helicity is the only finite-type topological invariant of magnetic fields

Let a magnetic field $\mathbf{B}$, $\mathrm{div}(\mathbf{B}) = 0$ has a support inside the unit ball $D \subset \mathbb{R}^3$. Denote by Ω the vector space of magnetic fields, equipped with C^∞ topology. Let $SDiff$ be the group of diffeomorphisms of D preserving the volume element. The standard action

$$A : SDiff \times \Omega \to \Omega$$

is well-defined. Let us call a continuous function

$$F : \Omega \to \mathbb{R}$$

a polynomial-type invariant of magnetic fields if the following conditions are satisfied:

–1. F is invariant with respect to A.

–2. $F(\mathbf{B})$ is defined as the restriction of an r-polylinear function $\hat{F}(\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_r) \to \mathbb{R}$ onto the diagonal $\mathbf{B} = \mathbf{B}_1 = \mathbf{B}_2 = \dots = \mathbf{B}_r$.

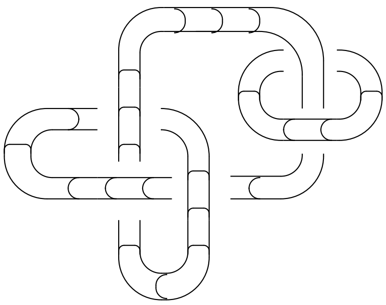
Example: Helicity quadratic invariant of magnetic fields $\mathbf{G}^2\mathbf{sm}^4$

Define the helicity of $\mathbf{B}$ by the formula:

$$\chi = \int (\mathbf{A}, \mathbf{B}) dD,$$

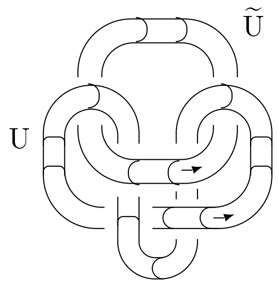
where $\mathbf{A}$ is the vector-potential of $\mathbf{B}$, $\mathrm{rot}(\mathbf{A}) = \mathbf{B}$, $\mathrm{div}(\mathbf{A}) = 0$, $\mathbf{A}(x) \to 0$, for $x \to \infty$.

Topological properties of helicity was discovered by M.Berger and G.Field (1984)



Theorem by S.S.Podkoritov (2004) (proof is easy)

An arbitrary finite-type invariant satisfies the equation: $F(\mathbf{B}^{Wh}) = 0$, where $\mathbf{B}^{Wh}$ is a vector field, which is a model for the Whitehead link:



Helicity is the asymptotic Hopf invariant (V.I.Arnol'd (1974))

Define the Gaussian linking number of two trajectories of $\mathbf{B}$ issuing from x_1 and x_2 for the large value T of time:

$$\Lambda(T; x_1, x_2) = \frac{1}{4\pi T^2} \int_0^T \int_0^T \frac{\langle \dot{x}_1(t_1), \dot{x}_2(t_2), x_1(t_1) - x_2(t_2) \rangle}{\| x_1(t_1) - x_2(t_2) \|^3} dt_1 dt_2,$$

where $x_i(t_i) = g^{t_i}(x_i)$, $i = 1, 2$ are the trajectories of the point x_i, given by the flow of $\mathbf{B}$, $\dot{x}_i(t_i) = \frac{d}{dt_i} g^{t_i} x_i$ are corresponding velocity vectors.

The following formula is satisfied:

$$\chi = \lim_{T \to +\infty} \int \int \Lambda(T; x_1, x_2) dx_1 dx_2.$$

Poincaré recurrence theorem

Let $\{g^t : \Omega \to \Omega\}$ is a flow generated by a magnetic field $\mathbf{B}$. For an (almost arbitrary) point $x \in \mathbb{R}^3$ and for an arbitrary $\varepsilon > 0$, there exists a real $t_0 > 0$, $t_0 = t_0(x, \varepsilon)$, such that $\mathrm{dist}(g^{t_0}(x), x) < \varepsilon$.

Almost all trajectories of the magnetic field $\mathbf{B}$ are almost closed.

A non-formal definition of the helicity χ by means of asymptotic Hopf invariant

$$\chi = \int \int \Lambda(l_1, l_2) d\Omega d\Omega,$$

where Ω is the spectrum (the space of all magnetic lines) of the field $\mathbf{B}$, $l_1, l_2 \in \Omega$, $\Lambda(l_1, l_2) = \lim_{T \to +\infty} \Lambda(T; x_1, x_2)$, $x_i \in l_i$, $i = 1, 2$.

A problem by V.I.Arnol'd

V.I.Arnol'd has formulated the following problem (1984-12): „To transform an asymptotic ergodic definition of the Hopf invariant for a divergence-free vector field to a theory by S.P.Novikov to generalize the Whitehead product in homotopy group of spheres".

A strategy toward the problem by V.I.Arnol'd

1. To find-out a finite-type invariant I, which satisfies Asymptotic and Stability properties.

2. To test an integral expression of I for asymptotic invariant of magnetic fields.

Example: **Quadratic helicity** $\chi^{(2)}$ $\mathbf{G^4sm^5}$

$$I(L_1, L_2, L_3) = \frac{1}{3}[lk(L_1, L_2)lk(L_2, L_3)+lk(L_2, L_3)lk(L_3, L_1)+lk(L_3, L_1)lk(L_1, L_2)],$$

$$\chi^{(2)} = \int\int\int P(l_1, l_2, l_3)d\Omega d\Omega d\Omega,$$

$$P(l_1, l_2, l_3) = \frac{1}{3}[\Lambda(l_1, l_2)\Lambda(l_2, l_3) + \Lambda(l_2, l_3)\Lambda(l_3, l_1) + \Lambda(l_3, l_1)\Lambda(l_1, l_2)],$$

$$l_1, l_2, l_3 \in \Omega.$$

Main result: a higher asymptotic invariant μ $\mathbf{G^{12}sm^6}$

For $m = 3$ there exist an asymptotic invariant M of the degree 12, which is not a function of pairwise linking numbers of components.

The invariant M is skew-symmetric, if at least 2 of the 3 linking numbers of components are trivial then $M = 0$.

The upper $\overline{\mu}$ and the lower $\underline{\mu}$ asymptotic invariants of divergence-free vector fields with respect to volume preserved diffeomorphisms are well-defined.

In the case all trajectories of $\mathbf{B}$ are closed, we get: $\overline{\mu}_{\mathbf{B}} = \underline{\mu}_{\mathbf{B}}$, which is associated with M

Proof of the Main result

The invariant M is finite-type invariant. An integral formula of M is in

On a new integral formula for an invariant of 3-component oriented links, Journal of Geometry and Physics, 53 (2005) 180-196 (by the author).

All the terms in this integral formula, applying for $\mathbf{B}$, admit asymptotic limits $T \to +\infty$ (lower and upper). In the case the trajectories of $\mathbf{B}$ are closed the lower and the upper limits coincide.

Asymptotic combinatorial invariants

Two operations for framed oriented links

Let I be a finite-type invariant of classical oriented m-component links in $\mathbb{R}^n$ (a generalized Whitehead product). We shall give necessary conditions that I is a (higher) asymptotic invariant.

Let $(\mathbf{L},\xi)$ be an arbitrary m-component framed link. For an arbitrary integer $r \in \mathbb{Z}$ let us define a framed m-component link $r(\mathbf{L},\xi)$, the components of this link are defined by the replacement of the corresponding framed component (L_i,ξ_i) of the oriented framed link $(\mathbf{L},\xi)$, $i = 1,\dots m$ to the component $r(L_i,\xi_i)$, which is the standard $(r,1)$–time winding along L_i.

Let $(\mathbf{L},\xi;L_0)$ be an arbitrary framed $(m-1)$–component link with a marked component $L_0 \subset \mathbf{L}$. Let us define m–component framed link $(\mathbf{L},\xi;L_0)^{\uparrow}$. The $(m-2)$ components of the link $(\mathbf{L},\xi)$ are transformed by the identity. The marked framed component (L_0,ξ_0) of the link $\mathbf{L}$ is transformed to the pair of parallel framed components $(L^{\uparrow}_{0,1},\xi^{\uparrow}_{0,1};L^{\uparrow}_{0,2},\xi^{\uparrow}_{0,2})$, the first component coincides with L_0, the second is defined by a small shift of the component L_0 along the frame ξ_0.

For a $(m-1)$–component framed link $(L,\xi;L_0)$ with one marked component and an integer r let us define two framed m-component links: $r((\mathbf{L},\xi;L_0)^{\uparrow})$, $(r(\mathbf{L},\xi;L_0))^{\uparrow}$.

Finite-type asymptotic invariants of oriented links

Let us say that a finite-type invariant I for m-component links is an asymptotic invariant of the degree s (the degree of the invariant I is distinguished from its order), if the following two equations are satisfied:

Asymptotic property

$$I(r(\mathbf{L},\xi)) = r^s I(\mathbf{L}) + o(r^s).$$

Stability property

$$I(r((\mathbf{L},\xi;L_0)^{\uparrow})) = I((r(\mathbf{L},\xi;L_0))^{\uparrow}) + o(r^s),$$

where $o(r^s)$ is a polynomial of r of the degree less then s, coefficients of this polynomial depend only on the isotopy class of the framed link $(\mathbf{L},\xi)$.

Conway polynomial

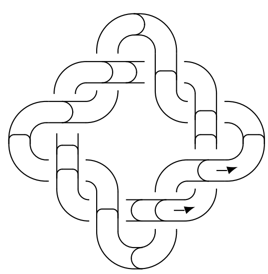
for m–component link $\mathbf{L}$:

$$\nabla_{\mathbf{L}}(z) = z^{m-1}(c_0 + c_1 z^2 + \cdots + c_n z^{2n}).$$

$m = 2$ $c_0(\mathbf{L})$ coincides with linking coefficient $lk(L_1, L_2)$.
$m = 1$ $c_1(\mathbf{L})$ is the Casson's invariants of knots.

2-component k-Hopf links $\mathbf{L}^+_{Hopf}(k)$, $\mathbf{L}^-_{Hopf}(k)$

The link $\mathbf{L}^+_{Hopf}(4) = (L_0,\xi_0)^{\uparrow}$, $lk(L_0,\xi_0) = 4$:



Generalized Sato-Levine invariant

The simplest invariant of 2-component links, discovered by M.Polyak and O.Viro (1994). This invariant is skew-symmetric, order 3.

Matveev S., Polyak M., *A simple formula for the Casson-Walker invariant*, Journal of Knot Theory and Its Ramifications, 18:6 (2009), 841–864;
arXiv:0811.0606.

The formula for the Generalized Sato-Levine

$$\beta(\mathbf{L}) = c_1(\mathbf{L}) - c_0(\mathbf{L})(c_1(L_1) + c_1(L_2)) - P(\mathbf{L}),$$

where

$$P(\mathbf{L}) = \frac{(k+1)k(k-1)}{6}, \quad c_0(\mathbf{L}) = k.$$

Ryo Nikkuni (2008), *Homotopy on spatial graphs and Generalized Sato-Levine Invariants*, arXiv:0710.3627v2.

In the case $c_0(\mathbf{L}) = 0$ we get $\beta(\mathbf{L}) = c_1(\mathbf{L})$ is the Sato-Levine invariant.

Relationship with Milnor's μ-invariants (Whitehead products)

Assume $\mathbf{L} = ((L_0, \xi_0), (L_1, \xi_1))$, $lk(L_0, \xi_0) = lk(L_1, \xi_1) = 0$, $lk(L_0, L_1) = 0$. The equation is satisfied:

$$\beta(\mathbf{L}) = \mu_{1,1,2,2}((L_0, \xi_0)^{\uparrow}, (L_1, \xi_1)^{\uparrow}).$$

Skein-relation for Generalized Sato-Levine invariant

For a homotopy of L_1 (or on L_2) with self-intersection point x the equation is satisfied:

$$\beta(\mathbf{L}_+) - \beta(\mathbf{L}_-) = O(x) lk(l_+, L_2) lk(l_-, L_2),$$

$O(x)$ is the sign of the self-intersection point x, l_+, l_- are the loops on L_1 (or on L_2).

Skein relation for Casson invariant

For a Δ-move of $\mathbf{L}$ with we get:

$$C_1(\mathbf{L}_+) - C_1(\mathbf{L}_-) = St_+ - St_-,$$

where St_+ (St_-) is the strangeness (in the sense of V.I. Arnol'd) of the projection of the knot $\mathbf{L}_+$ ($\mathbf{L}_-$) on the plane.

Generalized Sato-Levine invariant β and Casson invariant $C_1(\mathbf{L})$

Let $(\mathbf{L},\xi)$ be a framed knot. The following formula is satisfied:

$$\beta((\mathbf{L},\xi)^{\uparrow}) = 2lk(\mathbf{L},\xi)C_1(L). \tag{1}$$

Normalization

$$\beta(\mathbf{L}^{+}_{Hopf}(k)) = 0, \quad \beta(\mathbf{L}^{-}_{Hopf}(k)) = -P(k).$$

Generalized Sato-Levine invariant is not an asymptotic invariant

The following equation is satisfied:

$$I(r((\mathbf{L},\xi)^{\uparrow})) = Ar^4 + \dots, \quad I((r(\mathbf{L},\xi))^{\uparrow}) = Br^5 + \dots,$$

where the coefficients A, B depends only on $\mathbf{L}$. If $lk(\mathbf{L},\xi) = 0$, $B = 0$. Stability Property satisfies only in the case $lk(\mathbf{L},\xi) = 0$.

Proof: From Skein relation for the Generalized Sato-Levine invariant we get $s = 4$. From the Skein relation for the Casson invariant and the formula (1) we get $s = 5$.

Melikhov's invariant

γ is symmetric order 4 invariant of 3 component links.

$$\mathbf{L} = L_1 \cup L_2 \cup L_3,$$
$$\gamma(\mathbf{L}) = c_1(\mathbf{L}) -$$
$$((1,2)(2,3) + (2,3)(3,1) + (3,1)(1,2))(c_1(L_1) + c_1(L_2) + c_1(L_3))$$
$$-((3,1) + (2,3))(c_1(L_1 \cup L_2) - (1,2)(c_1(L_1) + c_1(L_2)))$$
$$-((1,2) + (3,1))(c_1(L_2 \cup L_3) - (2,3)(c_1(L_2) + c_1(L_3)))$$
$$-((2,3) + (1,2))(c_1(L_3 \cup L_1) - (3,1)(c_1(L_3) + c_1(L_1))),$$

where (i,j) are the linking numbers $k(L_i \cup L_j)$ of the pair of components L_i, L_j, $i,j = 1,2,3$, $i \neq j$, of the link $\mathbf{L}$ is defined.

Invariant $\tilde{M}$

$$\tilde{M}(\mathbf{L}) = (1,2)(2,3)(3,1)\gamma(\mathbf{L}) -$$
$$(1,2)^2(1,3)^2\beta(L_2 \cup L_3) + (2,3)^2(2,1)^2\beta(L_3 \cup L_1) + (2,3)^2(2,1)^2\beta(L_3 \cup L_1).$$

Properties of $\tilde{M}$

1. $\tilde{M}$ is skew-symmetric order 7 invariant for 3-component oriented links, which is not a function of the linking numbers of components.

2. If at least 2 of the 3 linking numbers $(1,2), (2,3), (3,1)$ are trivial, $\tilde{M} = 0$.

3. A perturbation

$$r(\mathbf{L}, \xi) \mapsto (r(\mathbf{L}, \xi))'$$

of a component of $r(\mathbf{L}, \xi)$ by means of an arbitrary r-strains braid keeps the invariant $\tilde{M}$, in particular $\tilde{M}(r(\mathbf{L}, \xi))$ is independent of framings. (Remark: The linking number $lk(L_1, L_2)$ of components of two-component links satisfies an analogous property.)

Asymptotic invariant M

Assuming that $(1,2)(2,3)(3,1) \neq 0$ define the invariant M by the formula:

$$M(\mathbf{L}) = \frac{\tilde{M}((2,3)L_1,(3,1)L_2,(1,2)L_3))}{(1,2)^4(2,3)^4(3,1)^4} + R((1,2),(2,3),(3,1)),$$

where $(\mathbf{L}) = (L_1, L_2, L_3)$, $R((1,2),(2,3),(3,1))$ is a suitable polynomial (providing the normalization), which depends only on linking numbers of the components.

Theorem

The invariant M is extended to a finite-type invariant for 3-component links without the assumption $(1,2)(2,3)(3,1) \neq 0$. This is an asymptotic invariant of the degree 12 with the normalization property:

$$M(\mathbf{L}^{+}_{Hopf}(k_1,k_2,k_3)) = 0,$$

where $\mathbf{L}^{+}_{Hopf}(k_1,k_2,k_3)$ is the simplest 3-component link with the given pairwise linking numbers of components.

Quadratic magnetic helicity $\chi^{(2)}$

Definition of $\chi^{(2)}$

Let $\mathbf{B}$ is a divergence-free vector field in D. Define $\Lambda^{(2)}$ by the formula:

$$\Lambda^{(2)}(T;x) = \frac{1}{T^2}(\int_0^T (\dot{x}(\tau), \mathbf{A})d\tau)^2,$$

where $x(\tau) = g^{\tau}(x)$ is a trajectory of $\mathbf{B}$ issuing from x, $\dot{x}(\tau) = \frac{d}{d\tau}g^{\tau}x = \mathbf{B}(x)$ is the corresponding vector of velocity. Define

$$\chi^{(2)} = \limsup_{T\to+\infty} \int \Lambda^{(2)}(T;x)dD.$$

Quadratic magnetic helicity $\chi^{(2)}$ is well-defined

By the Cauchy–Bunyakovsky–Schwarz inequality we get:

$$\Lambda^{(2)}(T;x) \le \frac{1}{T}\int (\dot{x}(\tau),\mathbf{A})^2 d\tau.$$

Therefore

$$\int \Lambda^{(2)}(T;x)dD \le \frac{1}{T}\int\int (\dot{x}(\tau),\mathbf{A})^2 d\tau dD = \int (\mathbf{B},\mathbf{A})^2 dD.$$

$$\int \Lambda^{(2)}(T;x)dD \le \delta^{(2)},$$

where

$$\delta^{(2)} = \int (\mathbf{B},\mathbf{A})^2 dD.$$

Quadratic magnetic helicity $\chi^{(2)}$ is an invariant

By the induction equation we get:

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{v}\times\mathbf{B} - \mathbf{grad} f,$$

where f is a function on U with a prescribed boundary conditions, which satisfies the equation

$$\Delta f = \mathrm{div}(\mathbf{v}\times\mathbf{B}).$$

The integral trajectory $x(\tau)$ of $\mathbf{B}$ is transformed into the trajectory $x'(\tau) = x(\tau) + dx(\tau) = x(\tau) + d(\mathrm{rot}(\mathbf{v}(x(\tau))\times\mathbf{B}(x(\tau))))$.

In each point of $x(\tau)$ the following equations are satisfied:

$$(\frac{\partial}{\partial t} + L_{\mathbf{v}})\mathbf{A} = \mathbf{grad} f,$$

$$(\frac{\partial}{\partial t} + L_{\mathbf{v}})\mathbf{B} = 0,$$

where $L_{\mathbf{v}}$ is the Lee derivative along the vector field $\mathbf{v}$, $\dot{x}(\tau) = \mathbf{B}(x(\tau))$.

The value $\Lambda^{(2)}$ is transformed by the following formula:

$$\Lambda^{(2)}(T;x) \mapsto \Lambda^{(2)}(T;x)+$$

$$\frac{2}{T^2}(\int_0^T(\mathbf{B}(x(\tau)),\mathbf{A}(x(\tau)))d\tau)(\int_0^T(\dot{x}(\tau),(\frac{\partial}{\partial t}+L_{\mathbf{v}})\mathbf{A}(x(\tau))d\tau \quad +$$

$$\int_0^T((\frac{\partial}{\partial t}+L_{\mathbf{v}})\mathbf{B}(x(\tau)),\mathbf{A}(x(\tau)))d\tau).$$

Therefore,

$$\Lambda^{(2)}(T;x) \mapsto \Lambda^{(2)}(T;x) + \int_0^T(\mathbf{B}(x(\tau)),\mathbf{grad}f(x(\tau)))dt.$$

To prove the invariance of $\chi^{(2)}$ it is sufficiently to prove that the transformation

$$\int \Lambda^{(2)}(T;x)dD \mapsto \int \Lambda^{(2)}(T;x)dD+$$

$$\frac{2}{T^2}(\int\int_0^T(\mathbf{B}(x(\tau)),\mathbf{A}(x(\tau)))dtdD)(\int\int_0^T(\mathbf{B}(x(\tau)),\mathbf{grad}f(x(\tau)))dtdD)+$$

$$\frac{1}{T^2}(\int\int_0^T(\mathbf{B}(x(\tau)),\mathbf{grad}f(x(\tau))dtdD)^2$$

is the identity for $T \to +\infty$. By the Newton-Leibniz theorem we get:

$$\int_0^T(\mathbf{B}(x(\tau)),\mathbf{grad}f(x(\tau)))dD = f(x(o)) - f(x(T)) \leq C,$$

where C depends on f, and is not depend on T. Therefore,

$$\int \Lambda^{(2)}(T;x)dD \mapsto \int \Lambda^{(2)}(T;x)dD + T^{-1}C_1,$$

where C_1 is bounded for $T \to +\infty$. Therefore we have:

$$\limsup_{T\to+\infty} \int \Lambda^{(2)}_{\mathbf{B}}(T;x)dD \mapsto \limsup_{T\to+\infty} \int \Lambda^{(2)}(T;x)dD.$$

The integral $\chi^{(2)}$ is an invariant with respect to volume-preserving diffeomorphisms.

Inequalities

$$\delta^{(2)} \geq \chi^{(2)} \geq \frac{\chi^2}{Vol(D)} \geq 0.$$

All values in this inequalities have the dimension $\mathrm{G}^4\mathrm{sm}^5$.

Geometrical meaning of quadratic magnetic helicity $\chi^{(2)}$

Example 1

Assume that a magnetic field $\mathbf{B}$ is localized inside the only flat thin magnetic tube $U \subset D$, all the trajectories of $\mathbf{B}$ are closed. This magnetic tube U is characterized by the following parameters:

– Φ is the magnetic flow trough the transversal cross-section of the tube,

– $\kappa \in \mathbb{Z}$ is the twisting coefficient of trajectories along the central axis of the tube (this twisting coefficient is integer and equals to the linking number of a pair of trajectories of $\mathbf{B}$)

– L is a length of the central line of the magnetic tube,

–Vol is the volume of the magnetic tube.

The magnetic energy is given by the expression:

$$U = \Phi^2 L,$$

The magnetic helicity is given by the expression:

$$\chi = \kappa\Phi,$$

The quadratic magnetic helicity is given by the expression:

$$\chi^{(2)} = \frac{\kappa^2\Phi^2}{Vol}.$$

Let us consider the following limit (the thickness of the magnetic tube tends to zero):

$$\kappa = const, \quad \Phi = const, \quad L = const, \quad Vol(L) \to 0.$$

Therefore the following equations are satisfied:

$$U = const, \quad \chi = const, \quad \chi^{(2)} \to +\infty.$$

Remark

For the given configuration of tubes the quadratic magnetic helicity gives no a lower bound of the magnetic energy.

Example 2

Assume that a magnetic field $\mathbf{B}$ is localized inside the pair of thin flat untwisted magnetic tubes $U_1 \cup U_2 \subset D$, all the trajectories of $\mathbf{B}$ inside each tubes are closed and unlinked. The following equation is satisfied:

$$\chi^{(2)} = (Vol(L_1) + Vol(L_2))^{-1}\chi^2.$$

Remark

For the given configuration of tubes the quadratic magnetic helicity gives a lower bound of the magnetic energy.

Application to the induction equation

The following equation is called the induction equation. This equation describes the evolution of magnetic field in a conductive liquid medium, assuming that the velocity field $\mathbf{v}$ of the medium is known:

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{rot}(\mathbf{v} \times \mathbf{B}) + \alpha \mathrm{rot}\mathbf{B} - \eta \mathrm{rotrot}\mathbf{B}. \tag{2}$$

Remark

The second term in the right side provides a growth of a magnetic field, this term is due to the helisity of the mean velocity field (the α-effect by Steenbeck, Krause, Rädler (1966)), or by neutrino (by D.D.Sokoloff and V.B.Semikoz (2004)). The third term in the right side provides a decrease of the magnetic field, this term is due to the magnetic dissipation.

Note that the induction equation (2) is not invariant and is not skew-invariant with respect to a mirror-symmetry. The magnetic energy is an invariant, the magnetic helicity is a skew-invariant with respect to a mirror symmetry.

The following well-known equations are satisfied:

$$\frac{d\chi}{dt} = -2\eta \int (\mathbf{B}, \mathrm{rot}\mathbf{B}) dD + 2\alpha \int (\mathbf{B}, \mathbf{B}) dD = -2\eta\chi^c + 2\alpha U,$$

where χ^c is called the current helicity, U is the magnetic energy.

Theorem ($\chi^{(2)}$ is a stable invariant)

Assuming the equation (2), the following inequalities are satisfied:

$$\frac{d\sqrt{\chi^{(2)}}}{dt} \le \eta \sqrt{\int (\mathrm{rot}\mathbf{B}, \mathbf{B})^2 dD} + \eta \sqrt{\int (\mathrm{rotrot}\mathbf{B}, \mathbf{A})^2 dD} +$$

$$\alpha \sqrt{\int (\mathbf{B}, \mathbf{B})^2 dD} + \alpha \sqrt{\int (\mathrm{rot}\mathbf{B}, \mathbf{A})^2 dD} +$$

$$\eta (\int (\mathrm{rotrot}\mathbf{B}, \mathrm{rotrot}\mathbf{B})^4 dD)^{1/8} (\int (\mathbf{A}, \mathbf{A})^2 dD)^{1/4} +$$

$$\alpha (\int (\mathrm{rot}\mathbf{B}, \mathrm{rot}\mathbf{B})^4 dD)^{1/8} (\int (\mathbf{A}, \mathbf{A})^2 dD)^{1/4},$$

where the right side of the equation is a limit of the corresponding difference ratio.

The inequality by V.I.Arnol'd

There exists a positive constant $C > 0$, which depends on the radius of the ball D, such that the following inequality is satisfied:

$$C^{-2}U^2(\mathbf{B}) \geq \chi^2(\mathbf{B}).$$

This inequality can be proved by means of the Fourier expansion of the magnetic field.

The expansion for magnetic field $\mathbf{B}$ is:

$$\mathbf{B} = \sum_k (\mathbf{c}_k^+ + \mathbf{c}_k^-) e^{\text{ß}\mathbf{k}x}, \tag{3}$$

where k is a number of a corresponding wave vector $\mathbf{c}_k^{\pm}$.

Variations on the theme of the Arnol'd inequality

Assuming that the magnetic field is given by a power spectrum:

$$c_k^{\pm} = \gamma^{\pm} k^{-\frac{\alpha}{2}}, \quad \alpha = \frac{5}{3}, \quad c_k^{\pm} = |\mathbf{c}_k^{\pm}|.$$

The expansion for the magnetic energy is:

$$E = \sum_k \mathbf{c}_k^+ \bar{\mathbf{c}}_k^+ + \mathbf{c}_k^- \bar{\mathbf{c}}_k^- = \sum_k |\mathbf{c}_k^+|^2 + |\mathbf{c}_k^-|^2.$$

The expansion for the magnetic helicity is:

$$\chi = \sum_k b_k^+ - b_k^-,$$

where all coefficients $b_k^{\pm}$ are non-negative.

Then we get:

$$c_k = |\mathbf{c}_k^+|^2 + |\mathbf{c}_k^-|^2 = (\gamma^+ + \gamma^-) k^{-\alpha},$$

$$b_k = b_k^+ - b_k^- = (\gamma^+ - \gamma^-) k^{-\alpha-1}.$$

The expansion for the square of the magnetic energy is

$$E^2 = \sum_k \frac{2(\gamma^+ + \gamma^-)^2}{\alpha - 1} k^{-2\alpha+1}.$$

The expansion for the square of the magnetic helicity is:

$$\chi^2 = \sum_k b_k^{(2)},$$

$$b_k^{(2)} = \frac{2(\gamma^+ - \gamma_-)}{\alpha} k^{-2\alpha-1}.$$

The expansion for the correlation tensor of the quadratic magnetic helicity is:

$$\delta^{(2)} = \sum_k d_k^{(2)}.$$

This gives an upper bound for the quadratic magnetic helicity:

$$d_k^{(2)} \leq \frac{\gamma^+ + \gamma^-}{\alpha^2} k^{-2\alpha}.$$

This gives an intermediate Fourier spectrum for $\delta^{(2)}$ (and therefor for the quadratic helicity $\chi^{(2)}$) with respect to the spectra of U^2 and χ^2.