Title: *Next order asymptotics for optimal transport and N-point configurations*

Abstract: We consider the sharp next-order asymptotics problems for: (1) the minimum energy for optimal N-point configurations; (2) the N-Marginal Optimal Transport; and (3) the Jellium problem for N-point configurations, in all three cases with Riesz costs with inverse power-law long-range interactions. The first problem describes the ground state of a Coulomb or Riesz gas, the second appears as a semiclassical limit of DFT energy, modelling a quantum version of the same system (and is called Uniform Electron Gas in the physicsliterature), and the third describes charges in a uniform negative background, a rough model for electrons in a metal. Recently the second-order terms in the large-N asymptotic expansions for power s in dimension d were shown for: (1) for $\max(0,d-2) \le s<d$ (remaining open outside this range prior to our paper, as previous methods break down); and for (2) for $0< s<d$. The asymptotics expansion for (3) has long been known for $s=d-2$, but it has been otherwise open until now.

In the present work, we extend the sharp asymptotics for: 1) to $0< s< \max(0,d-2)$; and for 3) to $0< s<d$. Our paper's unified proof for these sharp asymptotics for 0< s<d is based on a new and robust screening procedure, which allowed a series of improvements on the existing theory. Our methods and results are extendable to other potentials with long-range and short-range interaction.

Moreover, we show here for the first time that for inverse-power-law interactions with power $0<s<d$, the second-order terms for these three problems are equal. For the Coulomb cost in $d=3$, our result was the first to verify the physicists' long-standing conjecture regarding the equality of the second-order terms for Jellium and Uniform Electron Gas. Moreover, if the crystallization hypothesis in $d=3$ holds, which is an extension of Abrikosov's conjecture originally formulated in $d=2$, then our result was the first to verify the physicists' conjectured 1.4442 lower bound on the famous Lieb-Oxford constant. Our work rigorously confirmed some of the predictions formulated by physicists, regarding the optimal value of the Uniform Electron Gas second-order asymptotic term.

We also present here some asymptotic properties for the Uniform Electron Gas second-order constant $C(s,d), 0<s<d$.