The Geometry of 4-Manifolds:

Curvature in the Balance

I

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Centro di Ricerca Matematica Ennio De Giorgi,
Pisa, Italia. Il 8 giugno 2022.

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— A. Einstein, to G. Gamow

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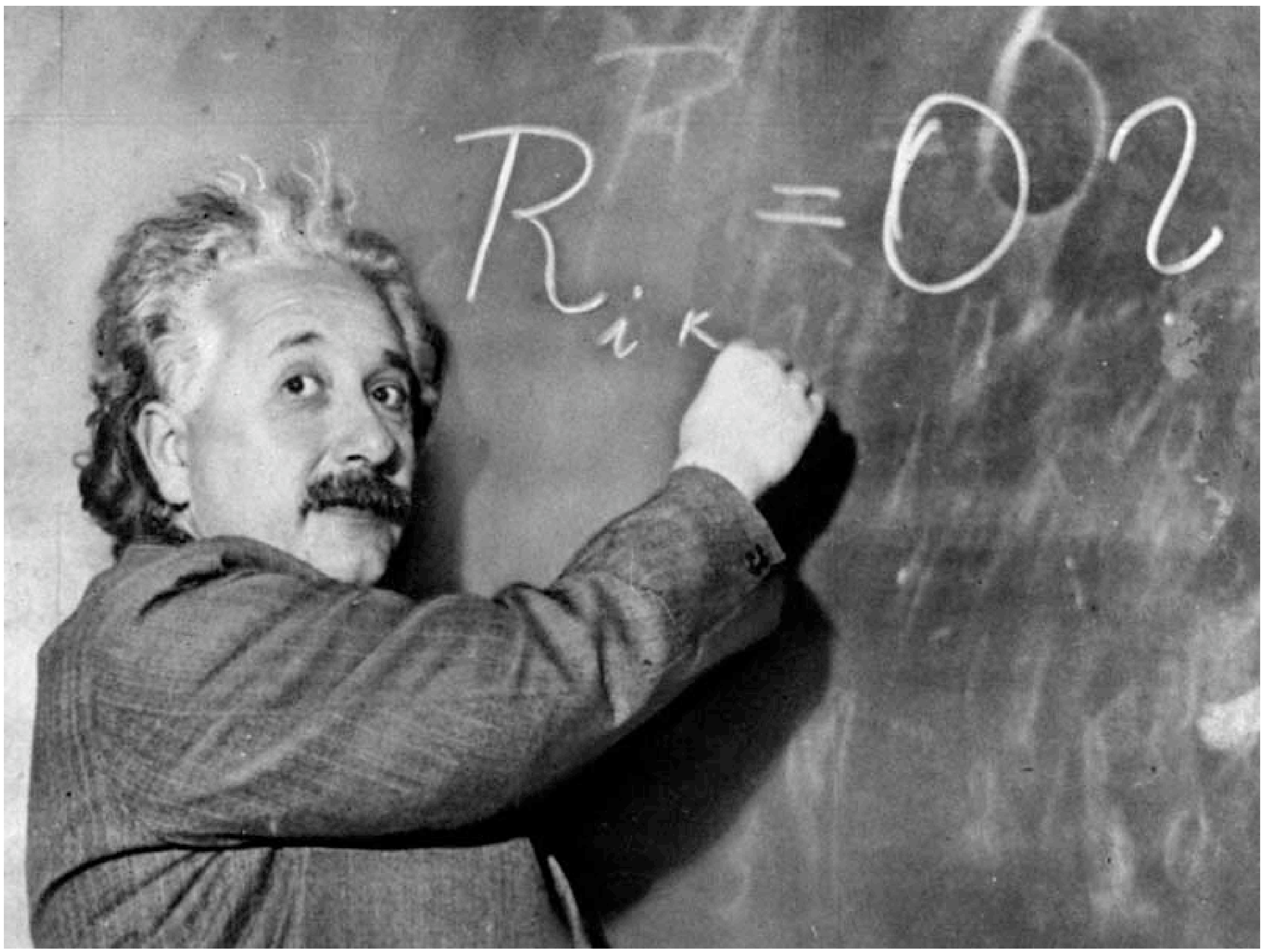

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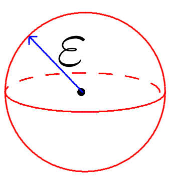
$$\frac{\mathrm{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s\,\frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



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Try to find Einstein metrics by minimizing?

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is realized by an Einstein metric g_j with $\lambda < 0$.

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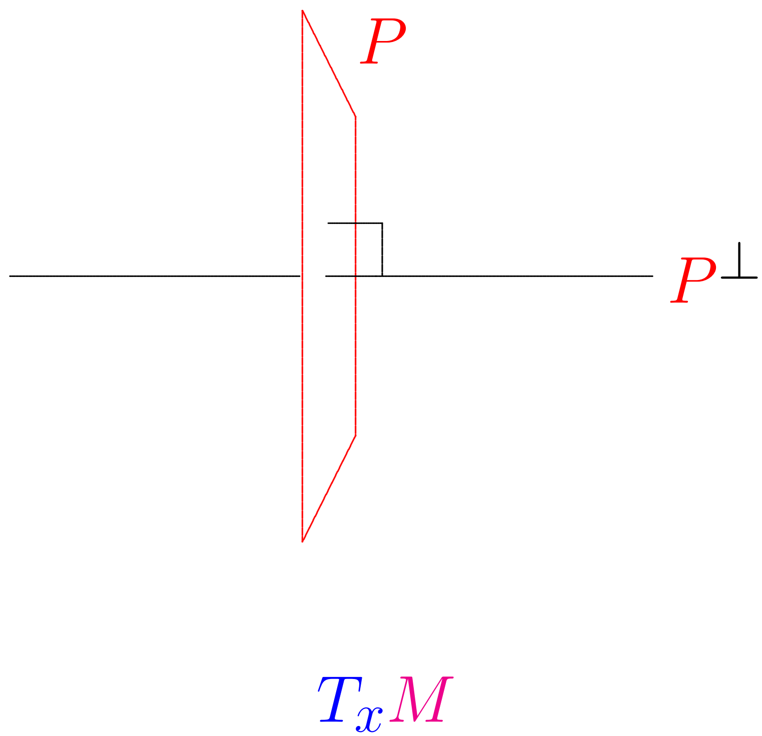

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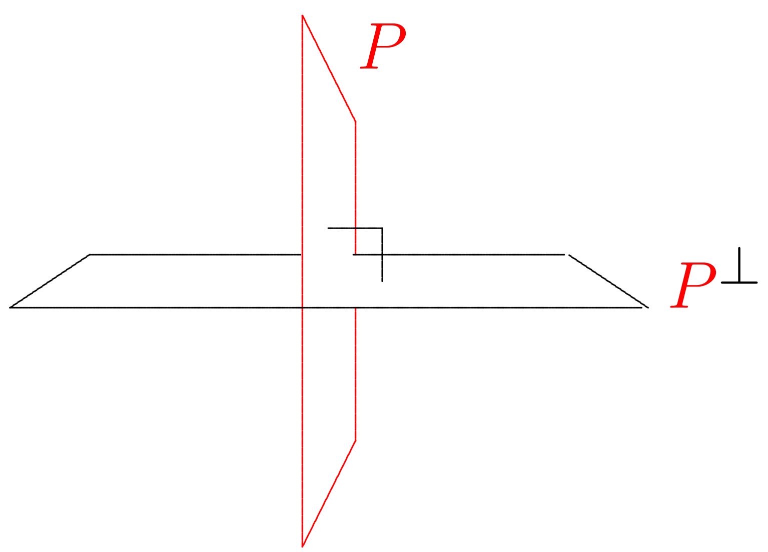

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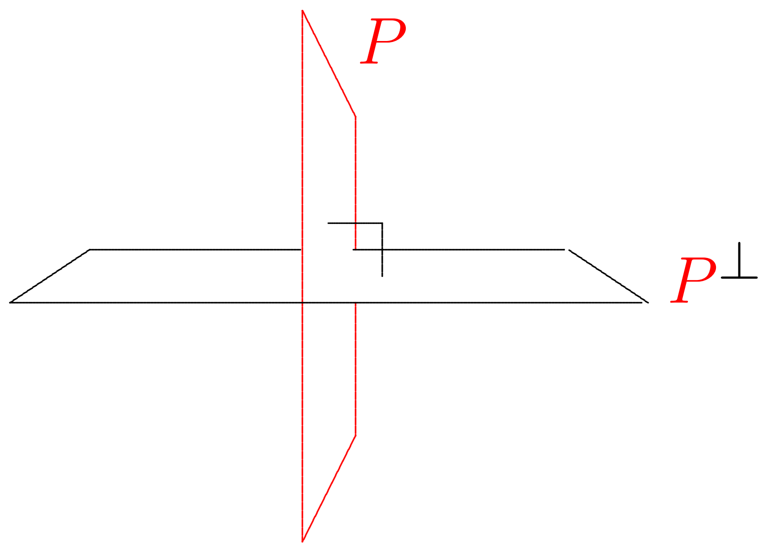

$$K(P) = K(P^{\perp})$$

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Integrals give four scale-invariant functionals.

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However, these are not independent!

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Euler characteristic

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu$$

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e.g. critical for self-dual Weyl functional

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"Curvature in the Balance."

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Hodge theory:

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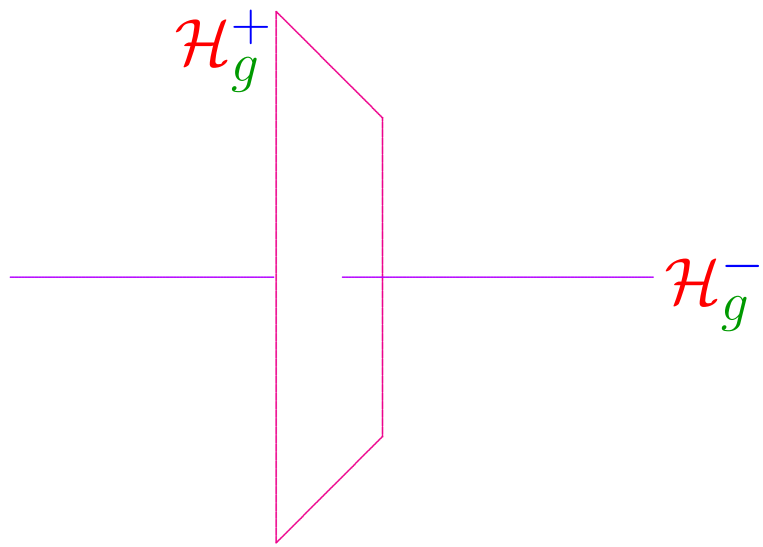

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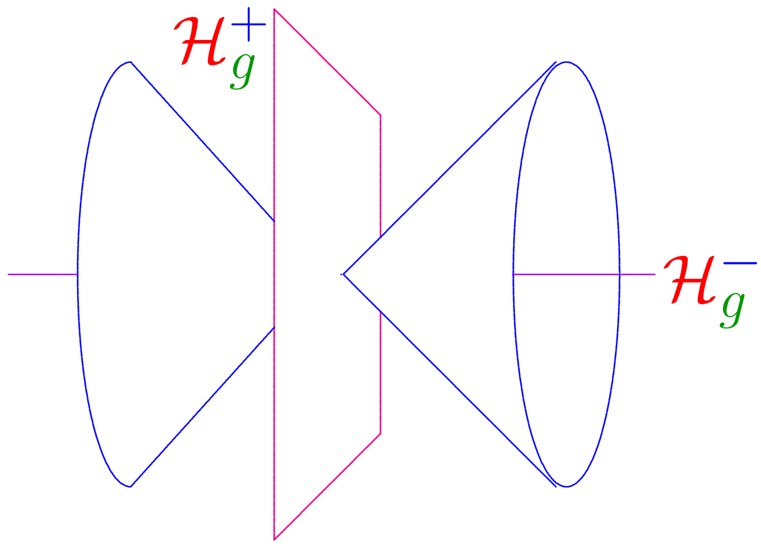

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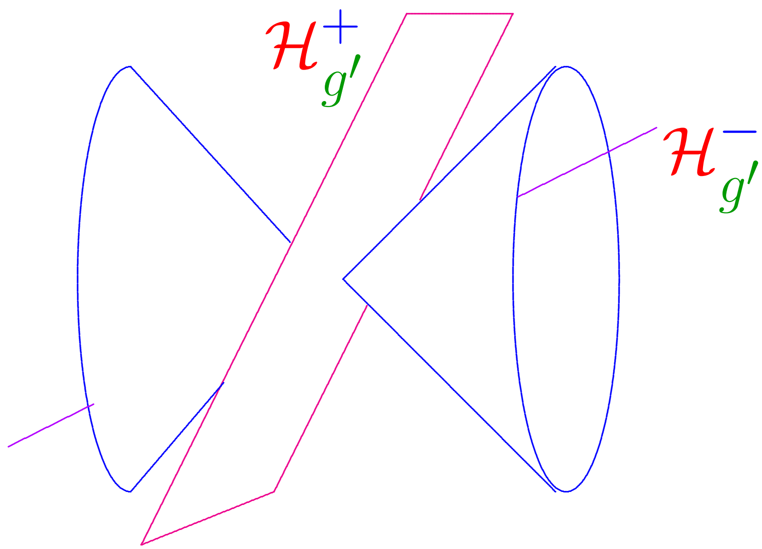

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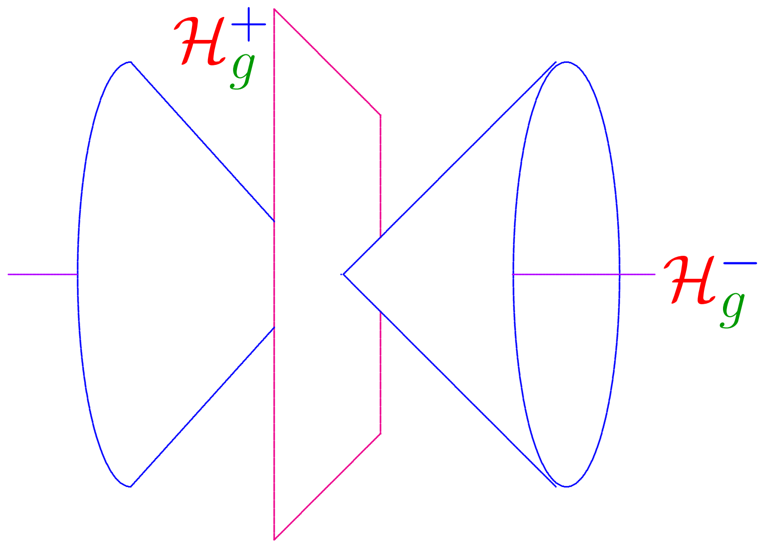

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Euler characteristic

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu$$

Signature

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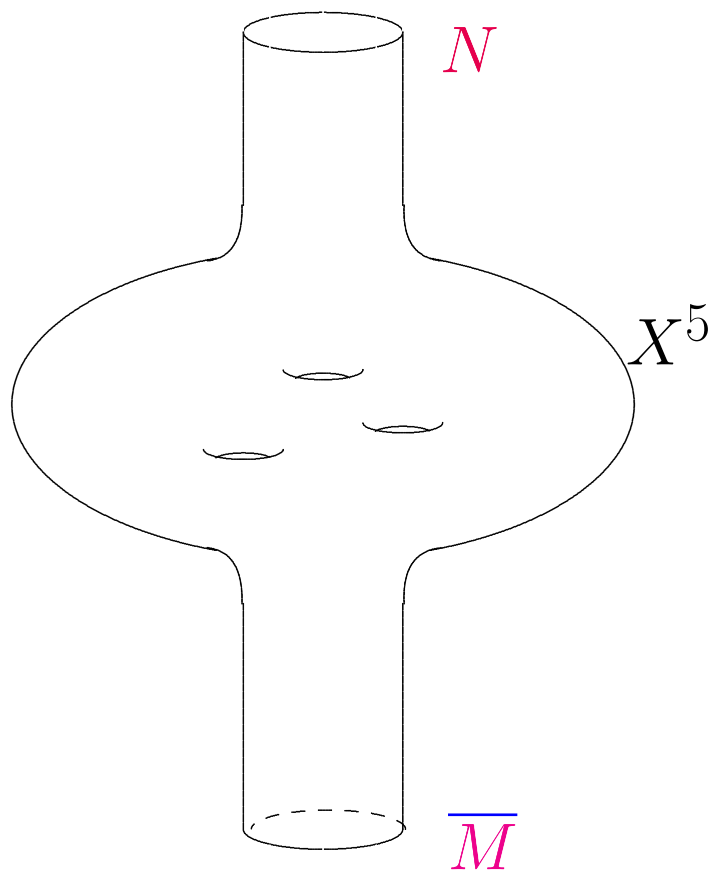
N
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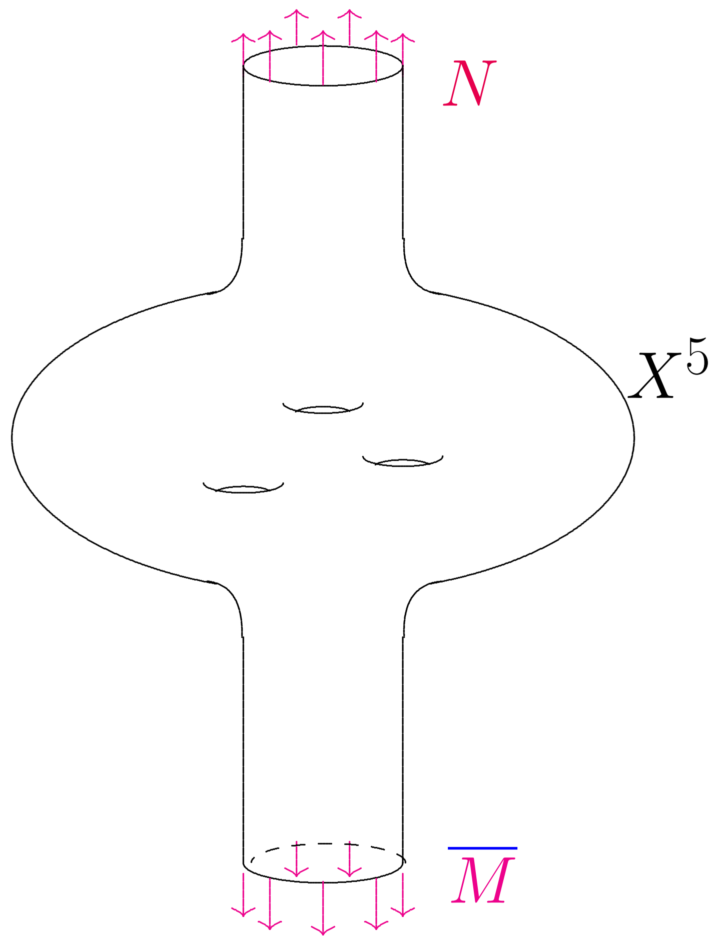
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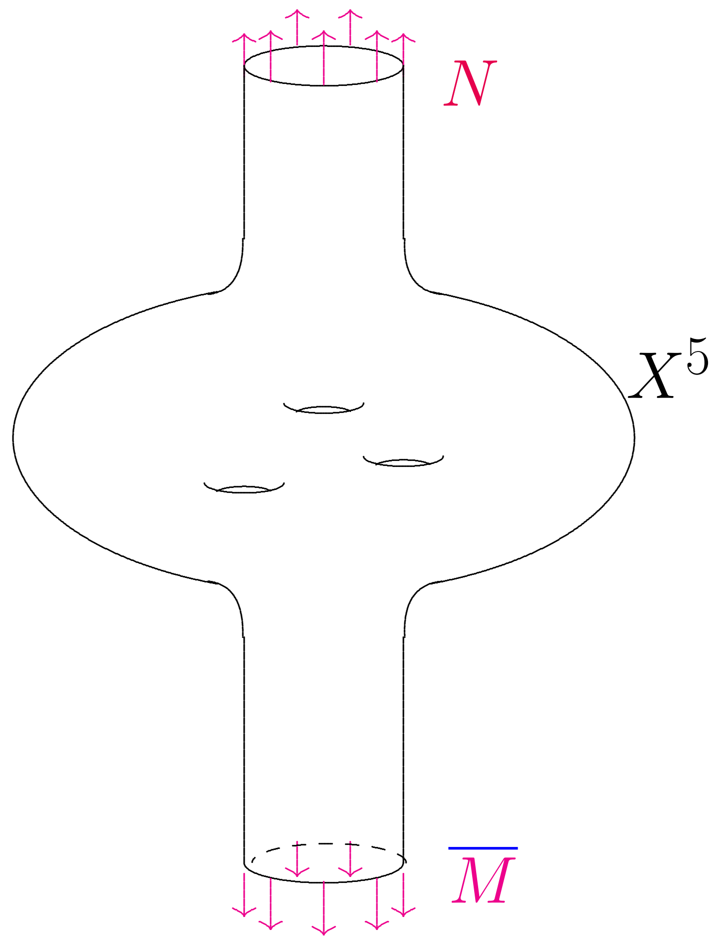

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Similarly, any quadratic curvature functional amounts to a linear combination terms of

$$\int_M s^2 d\mu_g \quad \text{and} \quad \int_M |W_+|^2 d\mu_g \ .$$

In these lectures, we'll carefully study both of these.

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which is closely related to the Yamabe problem.

Better-Known Variational Problem

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$$\mathcal{G}_M = \{ \text{ smooth metrics } g \text{ on } M\}$$

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where $V = \mathrm{Vol}(M, g)$ inserted to make scale-invariant.

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Then restriction $\mathscr{E}|_\gamma$ is bounded below.

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Difficulty: $L^2_1 \hookrightarrow L^p$ bounded, but not compact.

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Has $s =$ constant.

Unique up to scale when $s \leq 0$.

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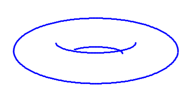
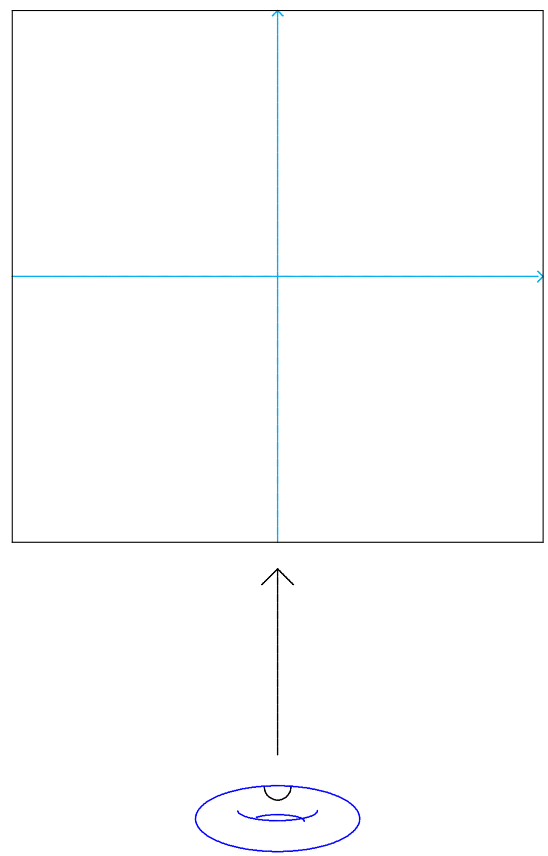
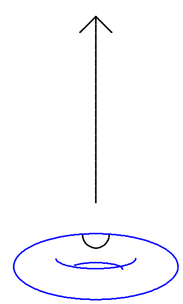
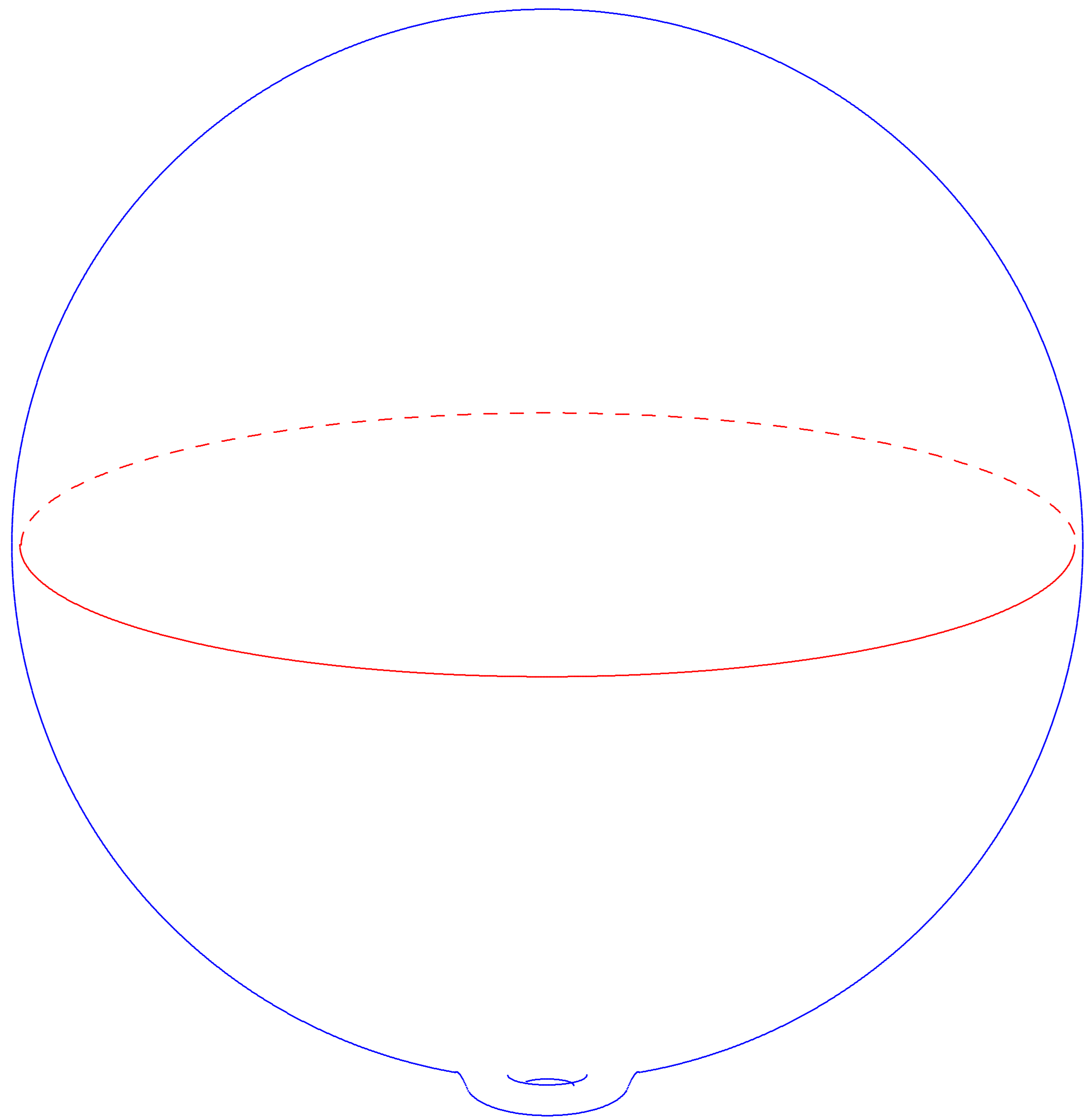




$$g_{jk} = \delta_{jk} + O(|x|^2)$$





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Schoen:

$$= \quad \text{only for round sphere.}$$

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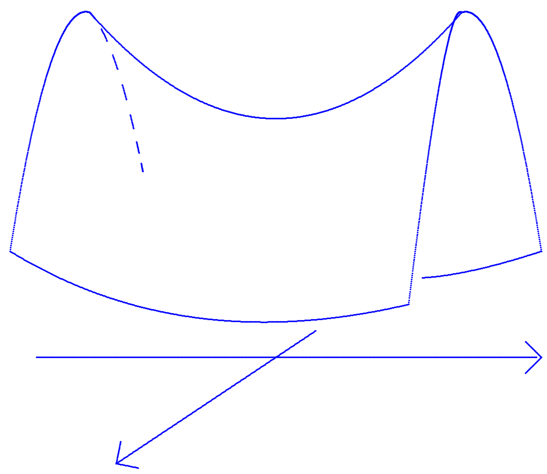

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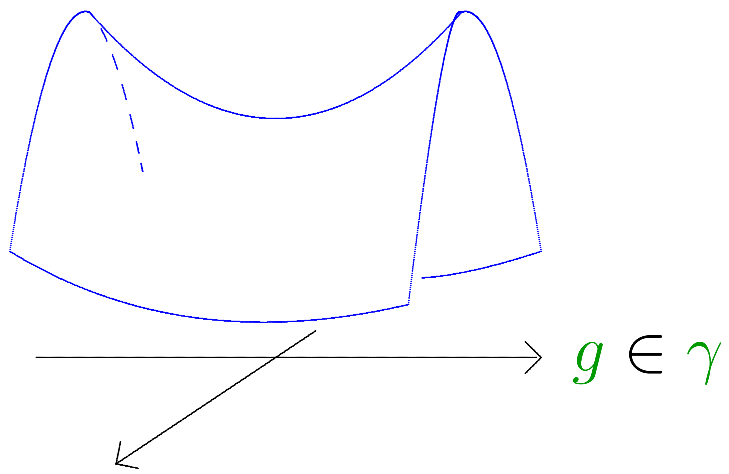

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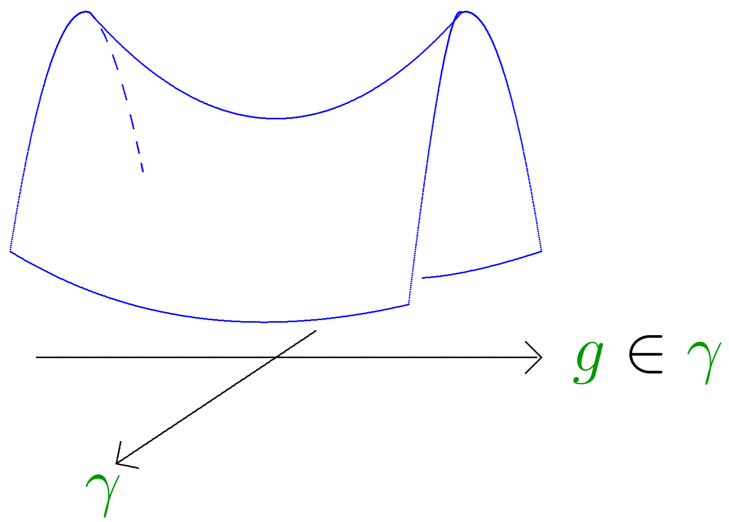

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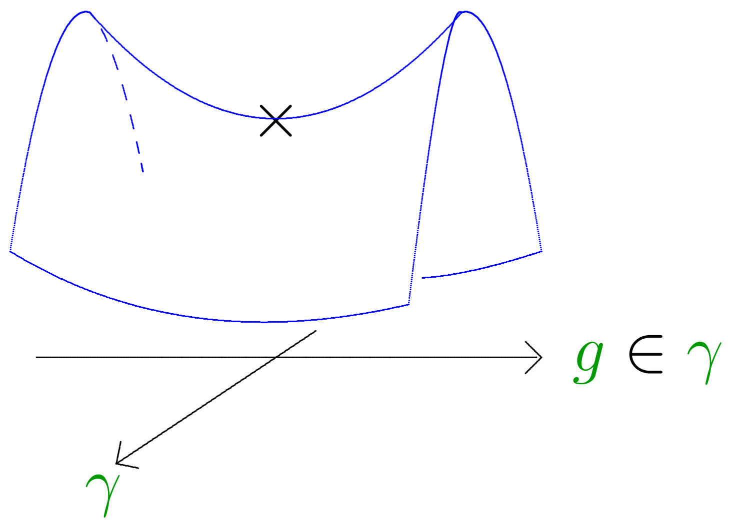

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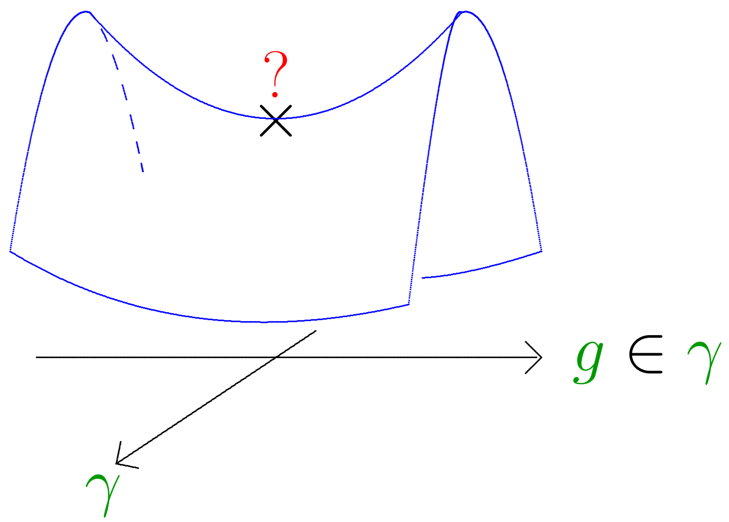

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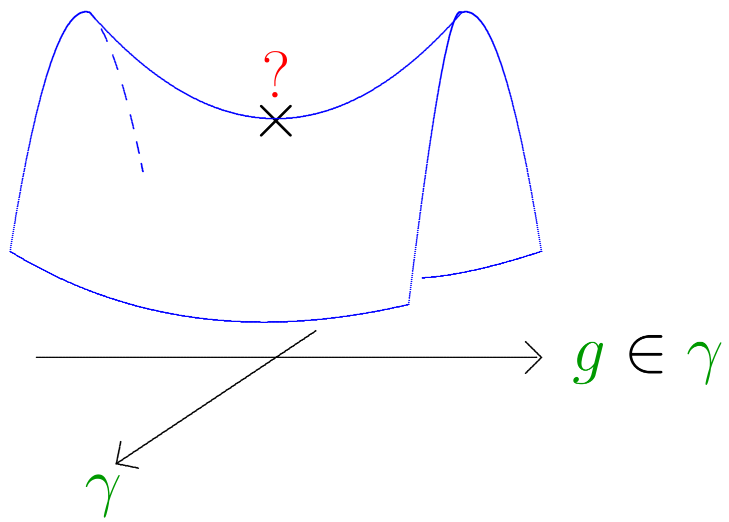

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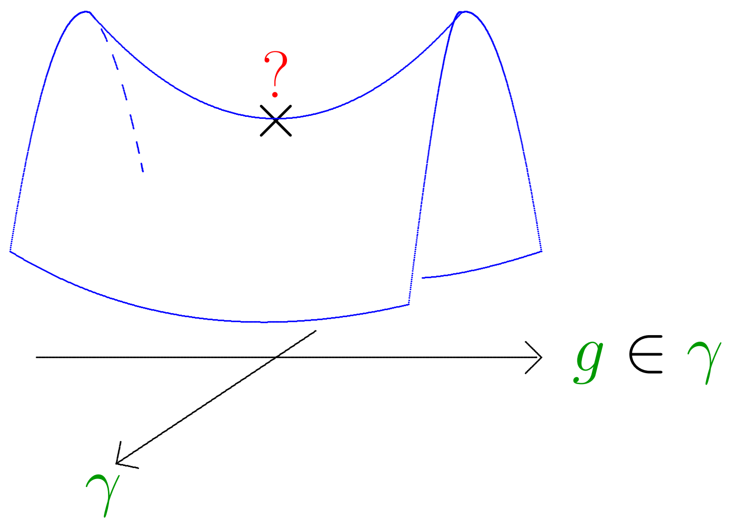

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But gives rise to a smooth-manifold invariant. . .

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R. Schoen ('87): "sigma constant"
O. Kobayashi ('87): "mu invariant"

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Seiberg-Witten theory

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"Deep triviality" $\Longrightarrow$

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$$\mathcal{I}_s(M_j) = \inf_g \int_{M_j} |s_g|^2 d\mu_g$$

is realized by an Einstein metric g_j with $\lambda < 0$.

Intermission

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This last result follows from…

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Kähler-Einstein means that (M, g) is Einstein, with almost-complex structure J s.t. $\nabla J = 0$ w/r to g.

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Corollary.

$$\mathscr{Y}(\mathbb{CP}_2) = 12\pi\sqrt{2} < 8\pi\sqrt{6} = \mathscr{Y}(S^4).$$

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Original proof used perturbed SW equations.

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Shows some other 4-mfds have $0 < \mathscr{Y}(M) < \mathscr{Y}(S^4)$,

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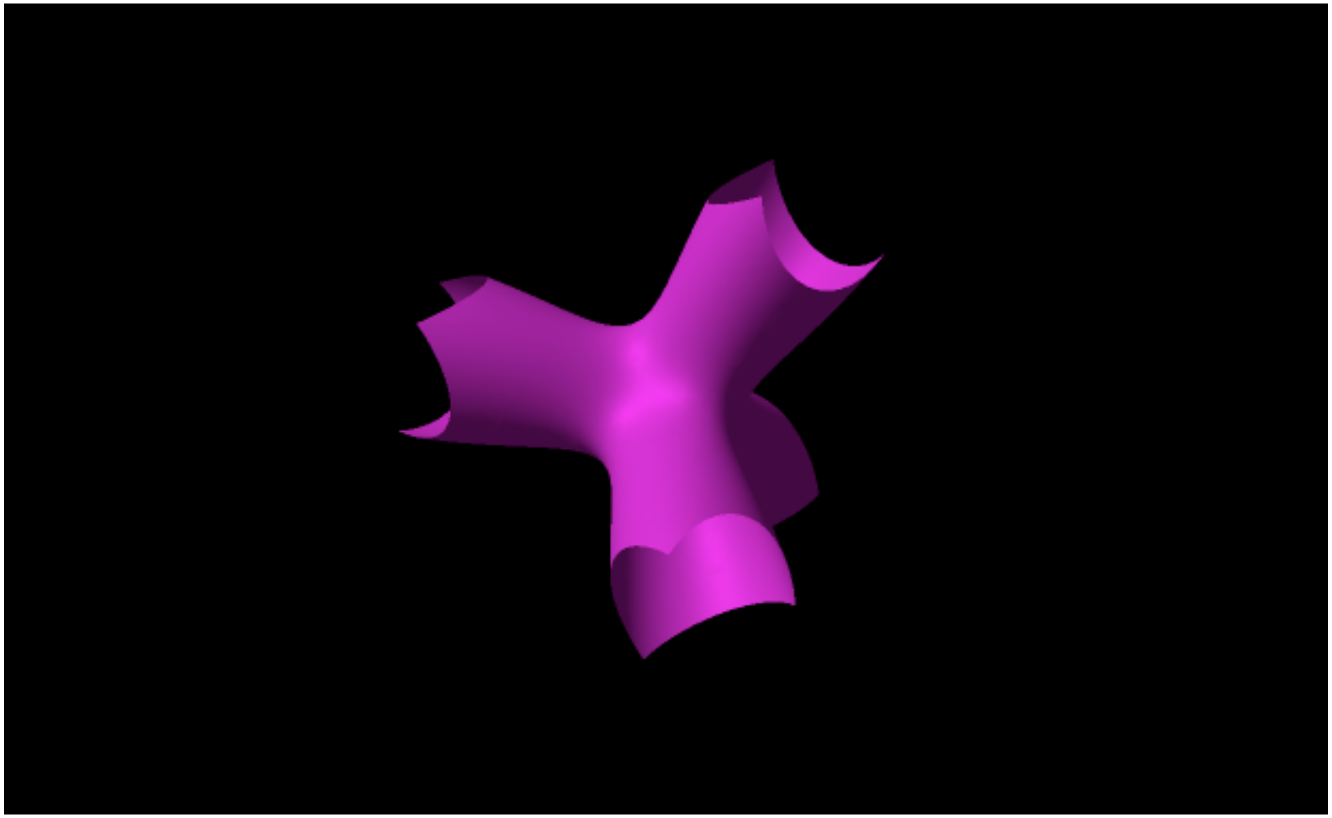

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Yau, Aubin, Siu, et al.

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≥ 5	"general type"	$-$	Yes

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These examples also show that the diffeomorphism invariant $\mathscr{Y}(M)$ is not simply a homeomorphism invariant — can detect “exotic” smooth structures.

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$$w_2 = 0 \qquad\qquad w_2 \neq 0$$

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where

$$\begin{aligned} k &= 1 + \frac{4}{3}m(m-1)(2m-1) \\ \ell &= \frac{2}{3}m(8m^2+1) \end{aligned}$$

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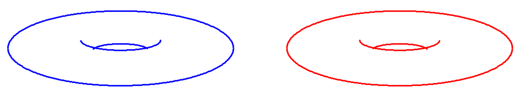

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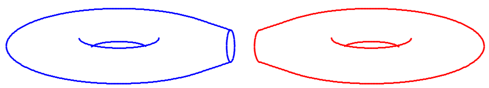

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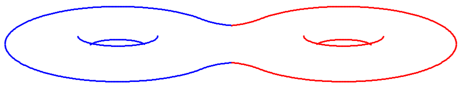

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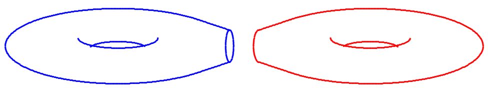

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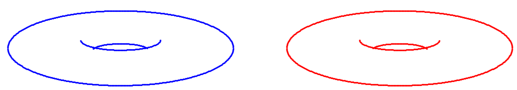

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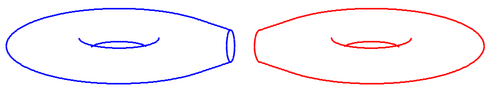

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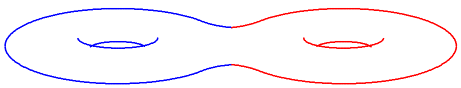

Example. Let $M \subset \mathbb{CP}_3$ a smooth hypersurface of degree ℓ. For concreteness:

$$x^\ell + y^\ell + z^\ell + w^\ell = 0$$

When $\ell = 2m + 1$ is odd, these are non-spin, so comparison of χ and τ shows homeomorphic to

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Non-linear, but elliptic once 'gauge-fixing'

$$d^*(\theta - \theta_0) = 0$$

imposed to eliminate automorphisms of $L \to M$.

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Bootstrapping with gauge-fixed equations, one gets L^p_k bounds for (Φ, θ) for all k, p.

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SW invariant $\in \mathbb{Z}_2$ means mod-2 mapping degree.

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Implies non-existence of metrics g for which $s > 0$.

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Basic strategy becomes: play several spin^c structures off against one another.

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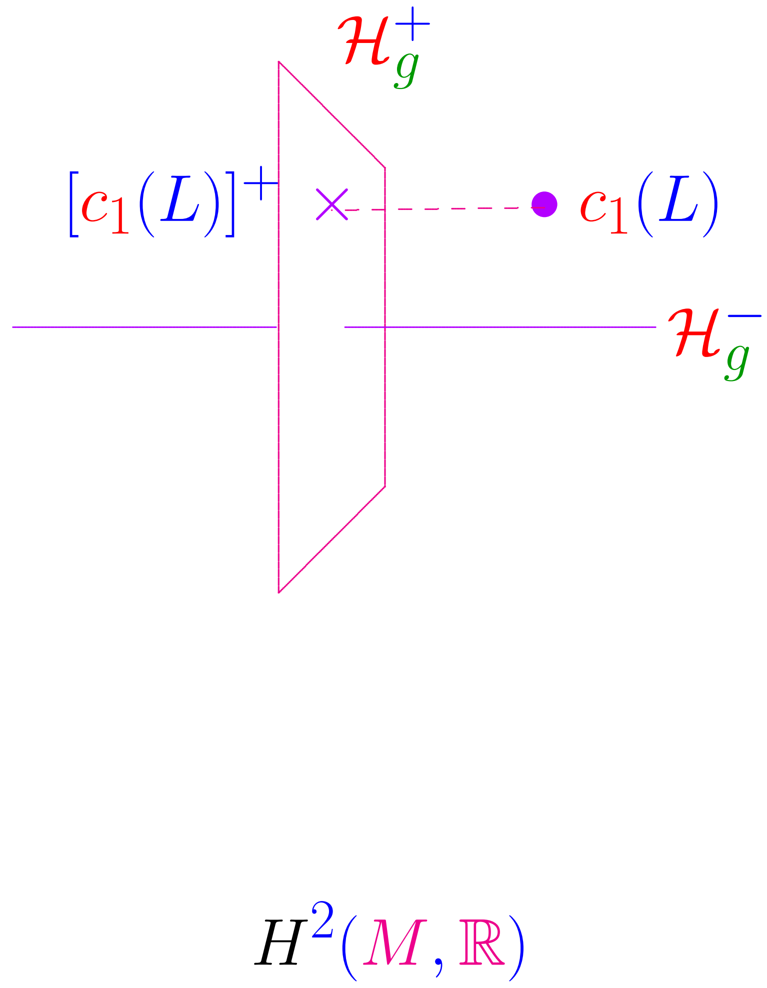
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This played an important role in the original proof, but is used only mildly in what follows.

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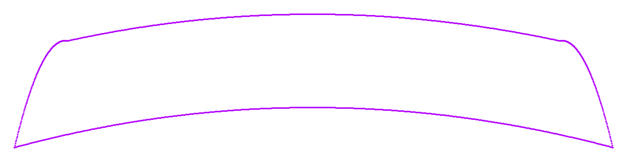

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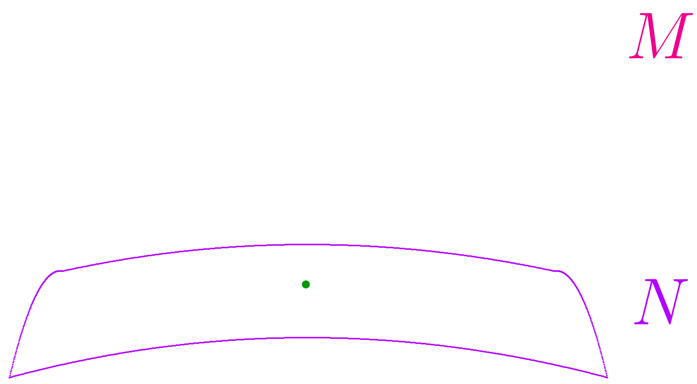

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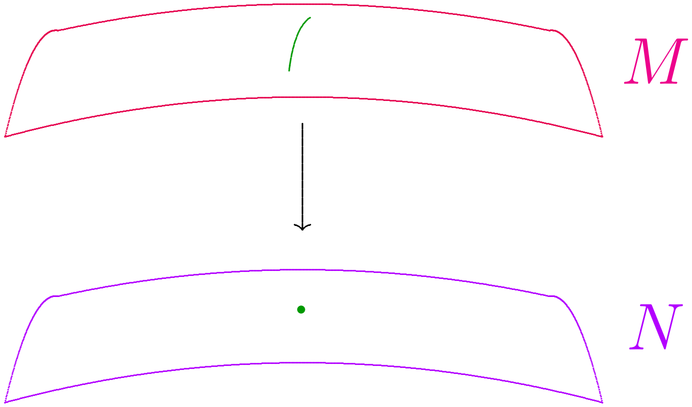

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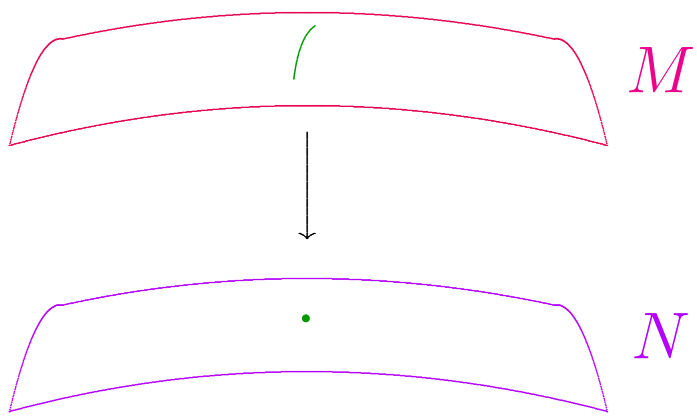

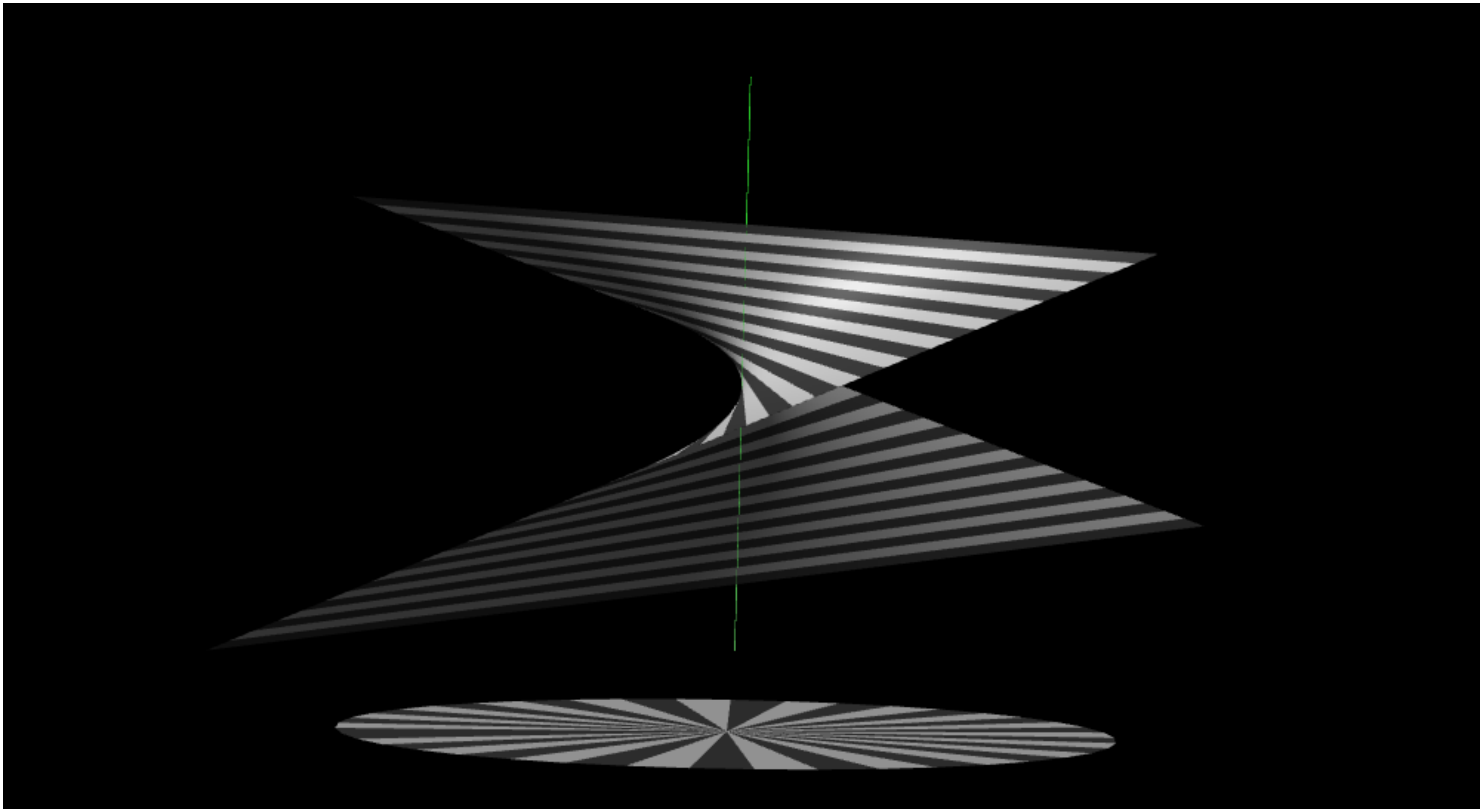

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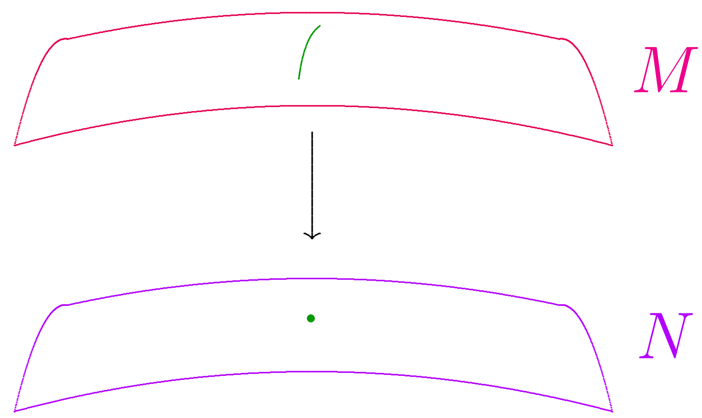

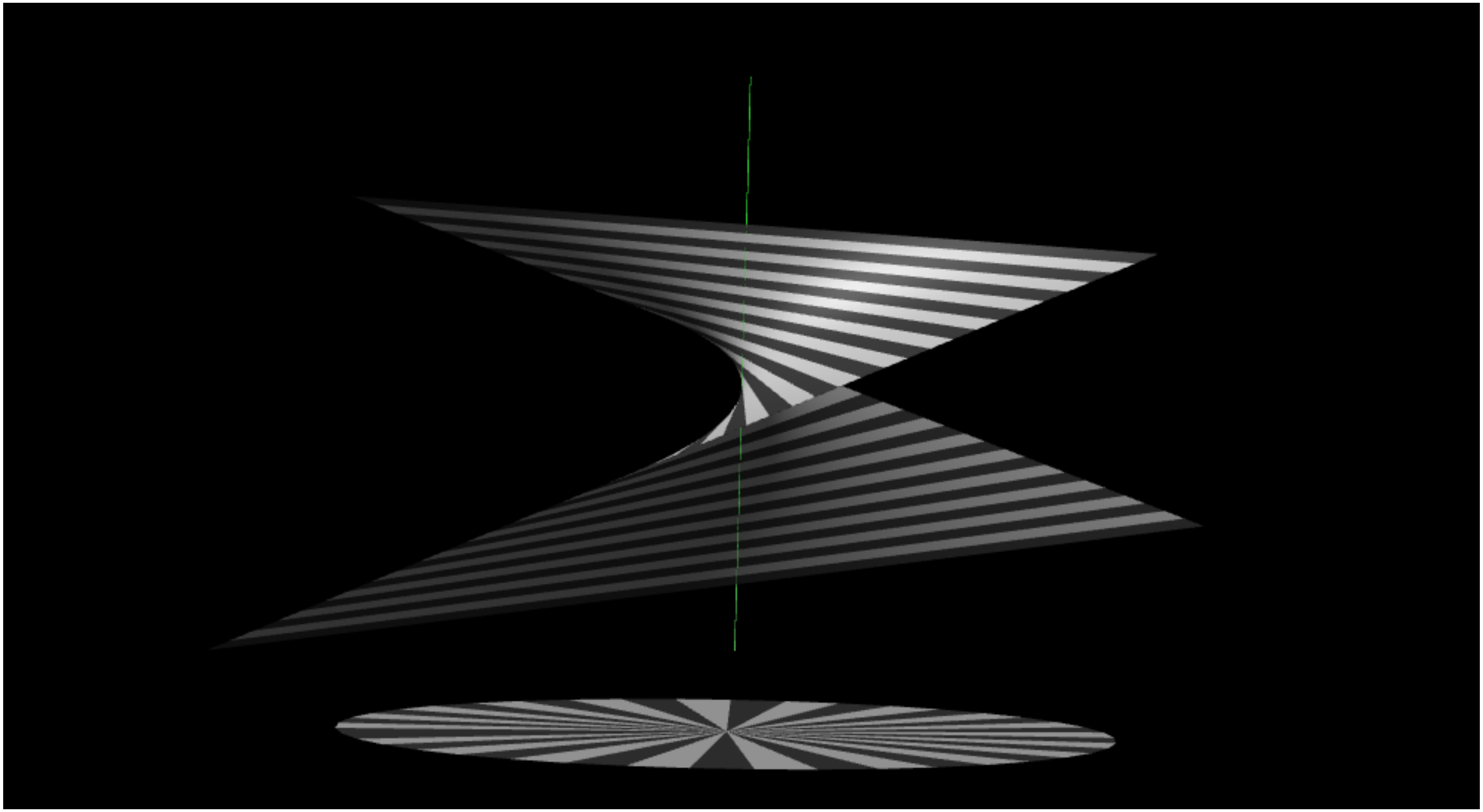

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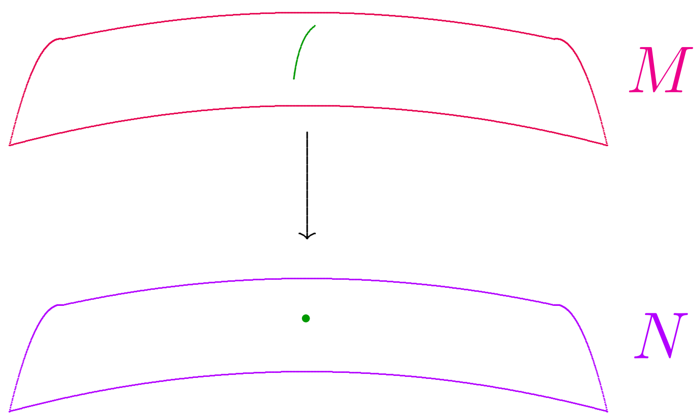

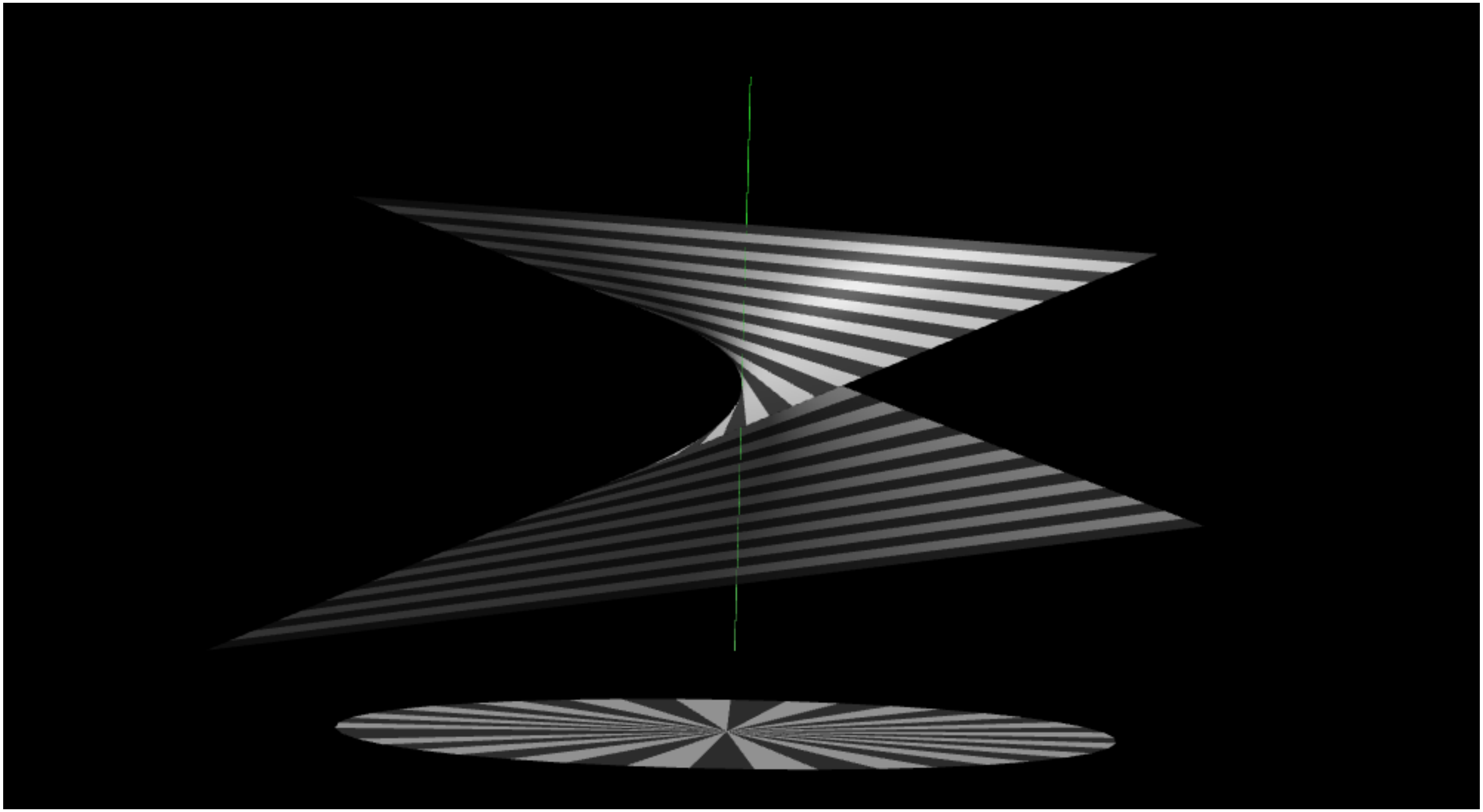

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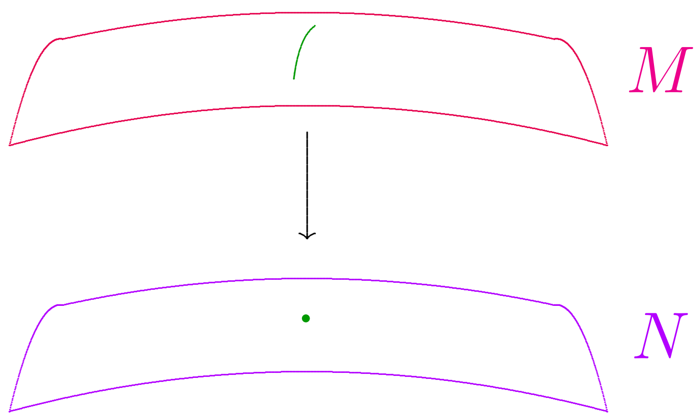

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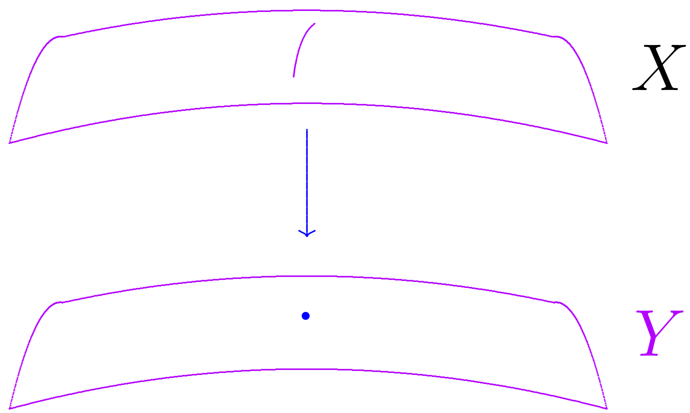

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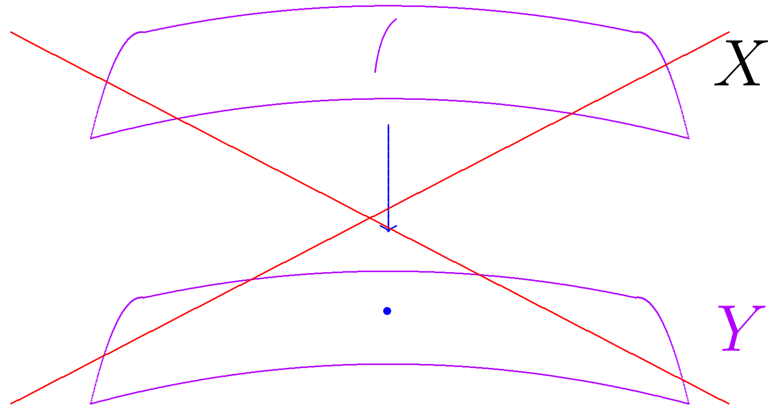

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In this setting, minimal model X is unique.

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Key ingredient: First Curvature estimate.

Next: how to use Second Curvature estimate.

First observe:

$$\frac{s^2}{24}+2|W_+|^2 = \frac{1}{27}\left[\left(s-\sqrt{6}|W_+|\right)^2+\frac{1}{8}\left(s+8\sqrt{6}|W_+|\right)^2\right]$$

First observe:

$$\frac{s^2}{24} + 2|W_+|^2 \quad \geq \quad \frac{1}{27}\left(s - \sqrt{6}|W_+|\right)^2$$

Hence:

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Equality forbidden, because would imply Kähler, but with wrong ratio of s^2 and $|W_+|^2$.

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Here one first shows generalized scalar curvature

$$\mathfrak{s} = s - \sqrt{6}|W_+|$$

would have to be constant if equality held.

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Einstein $\Longrightarrow$ $\frac{1}{3}{c_1}^2(X) > k$

Theorem. *Let* X *be a minimal surface of general type,*

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So being "very" non-minimal is an obstruction.

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Theorem (Aubin/Yau). *Compact complex manifold* (M^{2m}, J) *admits compatible Kähler-Einstein metric with* $s < 0 \iff c_1 < 0$.

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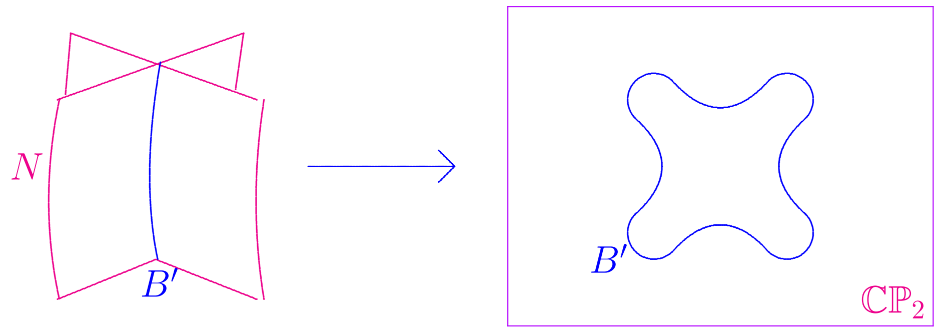

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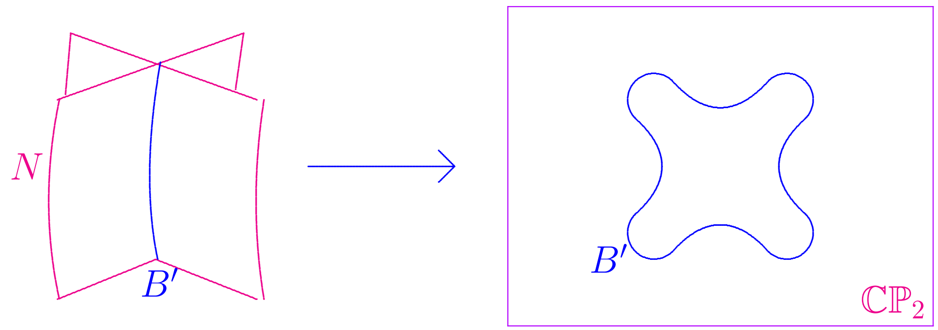
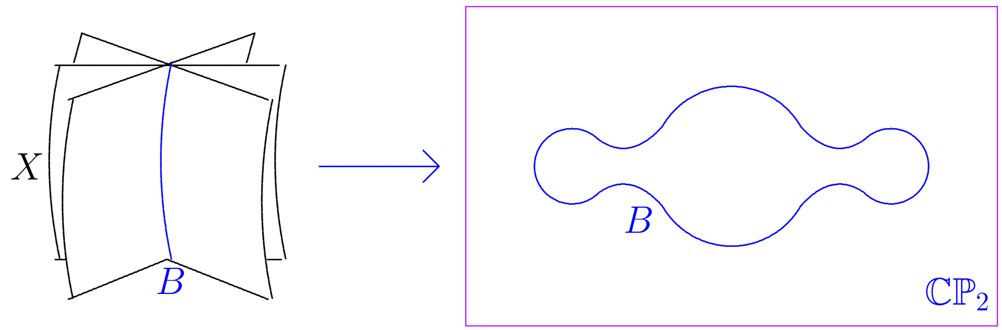


Aubin/Yau $\Longrightarrow$ N carries Einstein metric.

Now let X be a triple cyclic cover $\mathbb{CP}_2$, ramified at a smooth sextic



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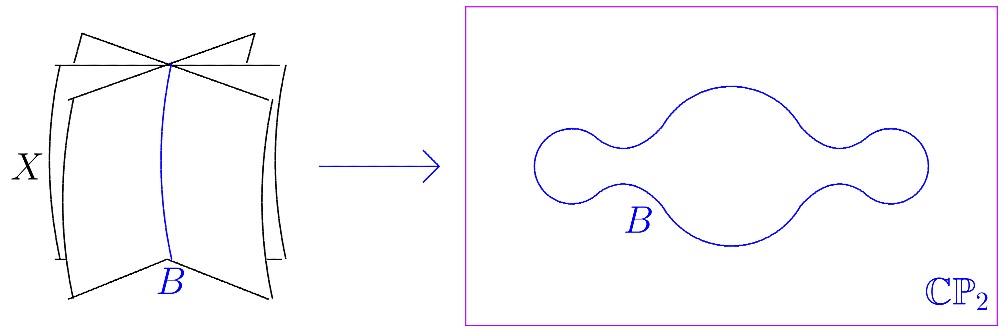

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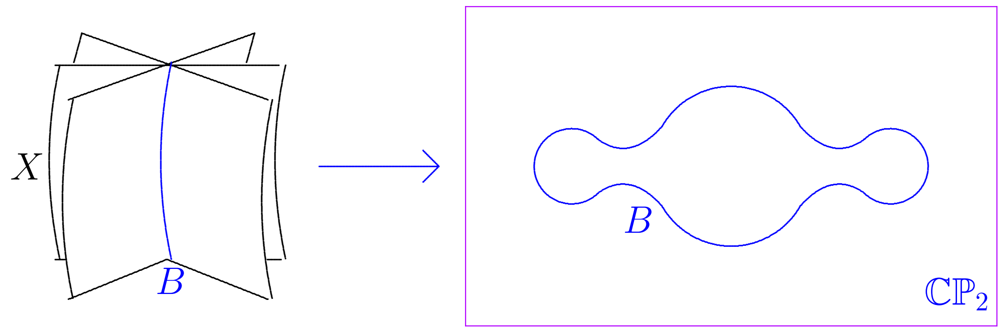

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Here

$$c_1{}^2(X) = 3$$

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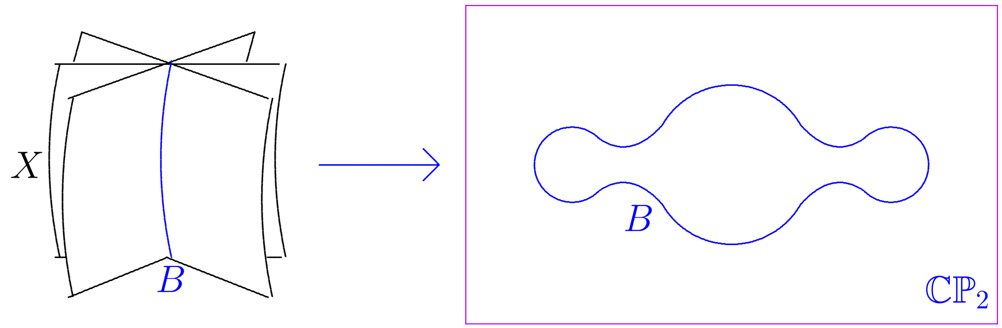
In example:

$$\begin{aligned} {c_1}^2(X) &= 3 \\ k = 1 &= {c_1}^2(X)/3 \end{aligned}$$

X is triple cover $\mathbb{CP}_2$ ramified at sextic



$$M = X \# \overline{\mathbb{CP}}_2.$$

Theorem $\Longrightarrow$ *no* Einstein metric on M.

But M and N are both
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Moral: Existence depends on diffeotype!

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End, Part I