Lecture 1: Distances between the conjugates of an algebraic number.

Let α be an algebraic number of degree $r \geq 2$. Denote the conjugates of α by $\alpha^{(1)}, \ldots, \alpha^{(r)}$. Let Σ be a subset of $\{1, \ldots, r\}$ of cardinality ≥ 2. By an elementary argument one can prove that

$$\prod_{\{i,j\} \subset \Sigma} |\alpha^{(i)} - \alpha^{(j)}| \geq 2^{-r(r-1)} M(\alpha)^{1-r}, \tag{1}$$

where $M(\alpha)$ denotes the Mahler measure of α. We are interested in the problem to obtain improvements of (1) with an exponent on $M(\alpha)$ larger than $1-r$. One may ask for such an improvement if one considers only algebraic numbers α generating a given number field K. More generally one may ask for such an improvement for all algebraic numbers α of given degree r. To be precise, we ask for improvements of (1) of the shape

$$\prod_{\{i,j\} \subset \Sigma} |\alpha^{(i)} - \alpha^{(j)}| \geq C(K) M(\alpha)^{1-r+\kappa} \quad \text{for every } \alpha \text{ with } \mathbf{Q}(\alpha) = K \tag{2}$$

with $\kappa > 0$, where K is a given number field of degree r and $C(K)$ is a constant depending only on K, or

$$\prod_{\{i,j\} \subset \Sigma} |\alpha^{(i)} - \alpha^{(j)}| \geq C(r) M(\alpha)^{1-r+\kappa} \quad \text{for every } \alpha \text{ of degree } r \tag{3}$$

with $\kappa > 0$, where $C(r)$ depends only on r.

As it turns out, such improvements do not exist if $r \leq 3$, so henceforth we assume that $r \geq 4$. In our lecture we will present some improvements of the shape (2), both with an ineffective constant $C(K)$ and with an effective constant $C(K)$ where in the latter case, κ is of course much smaller. Obtaining inequalities of the shape (3) seems as yet to be out of reach. If time permits, we will discuss the relation between inequalities of the shape (3) and certain other open problems and conjectures.

Lecture 2: Linear equations in unknowns from a multiplicative group in a function field (joint work with Umberto Zannier).

Let K be a field of characteristic 0, let Γ be a subgroup of the multiplicative group $(K^*)^n$ with coordinatewise multiplication and let $a_1, \ldots, a_n \in K^*$. We consider equations of the shape

$$a_1 x_1 + \cdots + a_n x_n = 1 \quad \text{in } \mathbf{x} = (x_1, \ldots, x_n) \in \Gamma. \tag{1}$$

It is natural to consider only *non-degenerate* solutions of (1), these are solutions such that $\sum_{i \in I} a_i x_i \neq 0$ for each non-empty subset I of $\{1, \ldots, n\}$. Schlickewei, Schmidt and the lecturer showed that if Γ is a group of finite rank r then (1) has at most $e^{(6n)^{3n}(r+1)}$ non-degenerate solutions. In this lecture I discuss a function field analogue of this result. More precisely, let k be an algebraically closed field of characteristic 0, and let K be an extension of k of transcendence degree ≥ 1 over k. Let Γ be a subgroup of $(K^*)^n$ containing $(k^*)^n$ and suppose that $\Gamma/(k^*)n$ has finite rank r. In this situation, (1) may have infinitely many non-degenerate solutions. We show that the set of non-degenerate solutions of (1) is contained in the union of at most $A(n,r) = \sum_{m=2}^{n+1} \binom{m}{2}^r - n + 1$ cosets of Γ modulo $(k^*)^n$. The proof rests on the fact that algebraic functions can be differentiated. The upper bound is obtained using Bezout's theorem from algebraic geometry.

This is an improvement of an earlier result by Mueller on equation (1). Our research builds further on work by Zannier alone on polynomial-exponential equations over function fields.