Hypoelliptic functional inequalities

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Preface

- Pseudo-differential operators and inequalities on nilpotent Lie groups.
- Pseudo-differential: brief mention of recent developments
  - Wigner transforms, Weyl systems, τ-quantizations on unimodular locally compact groups (Mantoiu+R., Doc Math. 2017, JGA 2018)
  - Symbolic calculus on $\mathbb{Z}^n$ (Botchway+Kibiti+R., JFA 2020), semiclassical, linear and nonlinear applications (Dasgupta+R.)
  - Weyl quantizations on $\mathbb{Z}^n$ and $\mathbb{T}^n$ (Esposito+R., Proc. R. Soc. Edinb.);
  - Pseudos on compact quantum groups (Akylzhanov+Majid+R., CMP 2018); Multipliers on locally compact groups (A.+R., JFA 2019)
  - Nonharmonic pseudo-differential operators (Tokmagambetov+R., Delgado+R., IMRN, JMPA, JdAM, 2016-2018)
- Hypoelliptic: inequalities and their hypoelliptic versions.
  - Based on works with (former & current) PhD students from Kazakhstan: Kassymov, Tokmagambetov, Sabitbek, Suragan, Yessirkegenov
  - Subelliptic Gevrey spaces, applications to weakly hyperbolic equations (with V. Fischer, and Chiara Taranto)
  - Background: our book with V. Fischer, Progress in Math., Birkhäuser, 2016; or even the book with Ville Turunen.

Introduction

In 1918, G. H. Hardy proved an inequality (discrete and in one variable) now bearing his name, which in $\mathbb{R}^n$ can be formulated as

$$\left\| \frac{f(x)}{\|x\|} \right\|_{L^2(\mathbb{R}^n)} \leq \frac{2}{n-2} \|\nabla f\|_{L^2(\mathbb{R}^n)}, \quad n \geq 3,$$

where ∇ is the standard gradient gradient in $\mathbb{R}^n$, $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$, $\|x\|$ is the Euclidean norm, and the constant $\frac{2}{n-2}$ is known to be sharp.

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Its L^p-version (used e.g. for p-Laplacian) takes the form

$$\left\| \frac{f(x)}{\|x\|} \right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p} \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad n \geq 2, \quad 1 \leq p < n,$$

where the constant $\frac{p}{n-p}$ is sharp.

A (connected and simply connected) Lie group $\mathbb{G}$ is graded if its Lie algebra:

$$\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i,$$

where $\mathfrak{g}_1, \mathfrak{g}_2, \ldots$, are vector subspaces of $\mathfrak{g}$, only finitely many not $\{0\}$, and

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \ \ \forall i, j \in \mathbb{N}.$$

If $\mathfrak{g}_1$ generates $\mathfrak{g}$ through commutators, the group is said to be stratified.
Example 1 (Abelian case). The abelian group $(\mathbb{R}^n, +)$ is graded: its Lie algebra $\mathbb{R}^n$ is trivially graded, i.e. $\mathfrak{g}_1 = \mathbb{R}^n$.
Example 2 (Heisenberg group). The Heisenberg group $\mathbb{H}_n$ is stratified: its Lie algebra $\mathfrak{h}_n$ can be decomposed as $\mathfrak{h}_n = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_1 = \oplus_{j=1}^n \mathbb{R}X_j \oplus \mathbb{R}Y_j$ and $\mathfrak{g}_2 = \mathbb{R}T$, where

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_t, \ \ Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_t, \ \ j = 1, \ldots, n, \ \ T = \partial_t. \tag{1}$$

Stratified groups/homogeneous Carnot groups

Let $\mathbb{G}$ be a stratified group, i.e. there is $\mathfrak{g}_1 \subset \mathfrak{g}$ (the first stratum), with its basis $X_1, \ldots, X_N$ generating $\mathfrak{g}$ through their commutators. Sub-Laplacian

$$\mathcal{L} := X_1^2 + \cdots + X_N^2$$

is hypoelliptic, $\nabla_H = (X_1, \ldots, X_n)$ is the horizontal gradient.
Folland (1972): $\mathcal{L}$ has a fundamental solution $= Cd(x)^{2-Q}$ for some homogeneous quasi-norm $d(x)$ called the $\mathcal{L}$-gauge.
☞ Fundamental solutions: Garofalo-Lanconelli (AIF 90 Heisenberg), Goldstein-Kombe (08 stratified), R.-Suragan (Adv. Math. 2017 boundary):

$$\left\| \frac{f(x)}{d(x)} \right\|_{L^2(\mathbb{G})} \leq \frac{2}{Q-2} \|\nabla_H f\|_{L^2(\mathbb{G})}, \quad Q \geq 3.$$

☞ Horizontal estimates: R.-Suragan (JDE, 2017): with Euclidean norm $|x'|$ on the first stratum, $x = (x', x'')$,

$$\left\| \frac{f(x)}{|x'|} \right\|_{L^2(\mathbb{G})} \leq \frac{2}{N-2} \|\nabla_H f\|_{L^2(\mathbb{G})}, \quad N \geq 3.$$

L^p-Hardy inequalities on stratified groups

☞ Fundamental solutions: Niu-Zhang-Wang'01, Adimurthi-Sekar'06, Danielli-Garofalo-Phuc'11, Jin-Shen'11:

$$\left\| \frac{f(x)}{d(x)} \right\|_{L^p(\mathbb{G})} \leq \frac{p}{Q-p} \|\nabla_H f\|_{L^p(\mathbb{G})}, \quad Q \geq 3, \quad 1 < p < Q.$$

☞ Horizontal estimates: R.-Suragan (JDE, 2017):

$$\left\| \frac{f(x)}{|x'|} \right\|_{L^p(\mathbb{G})} \leq \frac{p}{N-p} \|\nabla_H f\|_{L^p(\mathbb{G})}, \quad Q \geq 3, \quad 1 < p < N.$$

Both constants above are sharp. In particular, a special case of the horizontal estimate gives Badiale-Tarantello conjecture $(x = (x', x'') \in \mathbb{R}^N \times \mathbb{R}^{n-N})$

☞ Fractional: Ciatti-Cowling-Ricci (Adv. Math. 2015), quasi-norm $|\cdot|$,

$$\left\| \frac{f(x)}{|x|^\gamma} \right\|_{L^p(\mathbb{G})} \leq C_\gamma \|\mathcal{L}^{\gamma/2} f\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty, \quad 0 < \gamma < Q/p.$$

NB: this has been recently extended to $\gamma = Q/p$, to higher oder $\mathcal{L}$, and to $p \leq q$ (R.-Yessirkegenov 2018), with asymptotically best constants.

Beyond stratified groups

- graded group (but non-stratified) $\mathbb{G}$: may be no homogeneous (horizontal) gradient, sub-Laplacian or Laplacian. BUT: there are always so-called Rockland operators, which are left-invariant homogeneous hypoelliptic differential operators on $\mathbb{G}$.
- nilpotent (but not graded) groups: no invariant hypoelliptic differential operators $\Longrightarrow$ also no fundamental solutions.
  Recall DEF: $\mathcal{R}$ is hypoellipic if $\mathcal{R}f \in C^\infty$ implies $f \in C^\infty$.

Question is even: how to formulate Hardy inequality on

- (Q1) nilpotent Lie groups which are not graded?
- (Q2) graded groups which are not stratified?

Answers:

- (Q1) R.-Suragan, Hardy inequalities on homogeneous groups, Progress in Math., Vol. 327, Birkhäuser, 2019 (588pp)
- (Q2) R.-Yessirkegenov, Hypoelliptic functional inequalities, arxiv

Homogeneous groups (after Folland and Stein)

Actually, it can be shown that any (connected, simply connected) nilpotent Lie group is some $\mathbb{R}^n$ with a polynomial group law: $\mathbb{R}^n$ with linear group law, $\mathbb{H}_n$ with quadratic group law, etc.
So we can identify $\mathbb{G}$ with $\mathbb{R}^n$ (topologically).
If a Lie group (on $\mathbb{R}^n$) $\mathbb{G}$ has a property that there exist n-real numbers $\nu_1, ..., \nu_n$ such that the dilation

$$D_\lambda(x) := (\lambda^{\nu_1} x_1, ..., \lambda^{\nu_n} x_n), \quad D_\lambda : \mathbb{R}^n \to \mathbb{R}^n,$$

is an automorphism of the group $\mathbb{G}$ for each $\lambda > 0$, then it is called a homogeneous group.
Let us fix a basis $\{X_1, \ldots, X_n\}$ of the Lie algebra $\mathfrak{g}$ of the homogeneous group $\mathbb{G}$ such that X_k is homogeneous of degree ν_k.
Then the homogeneous dimension of $\mathbb{G}$ is

$$Q = \nu_1 + \cdots + \nu_n.$$

Homogeneous groups

A class of homogeneous groups is one of most general subclasses of nilpotent Lie groups, that is, the class of homogeneous groups gives almost the class of all nilpotent Lie groups but is not equal to it, Dyer70 gave an example of a (nine-dimensional) nilpotent Lie group that does not allow for any family of dilations.

Special cases of the homogeneous groups.

- the Euclidean group $(\mathbb{R}^n; +)$,
- Heisenberg groups,
- stratified groups (or homogeneous Carnot groups),
- graded Lie groups.

✎ Folland and Stein: Hardy spaces on homogeneous groups, Princeton Univ, Press, 1982.

✎ Fischer and Ruzhansky: Quantization on nilpotent Lie groups, Progress in Math., Birkhäuser, 2016.

✎ Ruzhansky and Suragan: Hardy inequalities on homogeneous groups, Progress in Math., Birkhäuser, 2019.

Hardy and Rellich inequalities on homogeneous groups

Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q, with a homogeneous quasi-norm $|\cdot|$. Then for all $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$, we have

Hardy:

$$\left\|\frac{f}{|x|}\right\|_{L^p(\mathbb{G})} \leq \frac{p}{Q-p}\left\|\mathcal{R}_{|x|}f\right\|_{L^p(\mathbb{G})}, \quad 1<p<Q,$$

Rellich:

$$\left\|\frac{f}{|x|^2}\right\|_{L^2(\mathbb{G})} \leq \frac{4}{Q(Q-4)}\left\|\mathcal{R}_{|x|}^2 f + \frac{Q-1}{|x|}\mathcal{R}_{|x|}f\right\|_{L^2(\mathbb{G})}, \quad Q \geq 5,$$

where $\mathcal{R}_{|x|} := \frac{d}{d|x|}$. Moreover, the constants above are sharp and are attained if and only if $f=0$.

M. Ruzhansky and D. Suragan.
Hardy and Rellich inequalities, identities, and sharp remainders on homogeneous groups.
Advances in Mathematics, 317:799–822, 2017.

Observations

☞ If $\mathbb{G}$ is (abelian) $\mathbb{R}^n$, and $|\cdot|$ is the Euclidean norm, this gives a refinement of the classical Hardy inequality:

$$\left\| \frac{f}{|x|} \right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p} \left\| \mathcal{R}_{|x|} f \right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p} \left\| \nabla f \right\|_{L^p(\mathbb{R}^n)} .$$

☞ Already in (abelian) $\mathbb{R}^n$, our inequality

$$\left\| \frac{f}{|x|} \right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p} \left\| \mathcal{R}_{|x|} f \right\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < n,$$

is new for any other non-Euclidean (quasi-norm), with sharp constant.

☞ Our idea: Consistently work with radial derivative $\mathcal{R}_{|x|} := \frac{d}{d|x|}$ and with the Euler operator $\mathbb{E} := |x| \frac{d}{d|x|}$.

☞ This yields many inequalities: Hardy, Rellich, Caffarelli-Kohn-Nirenberg, Sobolev type, ..., with best constants for any homogeneous quasi-norm.

Sobolev type inequalities through identities

Theorem

Let $\mathbb{G}$ *be a homogeneous group of homogeneous dimension* Q. *Then for all* $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ *and* $1 < p < Q$

$$\left\| \frac{p}{Q}\mathbb{E}f \right\|^p_{L^p(\mathbb{G})} - \|f\|^p_{L^p(\mathbb{G})} = p\int_{\mathbb{G}} I_p\left(f, -\frac{p}{Q}\mathbb{E}f\right)\left|f + \frac{p}{Q}\mathbb{E}f\right|^2 dx, \quad (2)$$

where $\mathbb{E} := |x|\frac{d}{d|x|}$ *is the Euler operator, and* I_p *is given by*

$$I_p(h, g) = (p-1)\int_0^1 |\xi h + (1-\xi)g|^{p-2}\xi d\xi.$$

M. Ruzhansky, D. Suragan and N. Yessirkegenov.
Sobolev type inequalities, Euler-Hilbert-Sobolev and Sobolev-Lorentz-Zygmund spaces on homogeneous groups.
Integral Equations and Operator Theory, 90:10, 2018.

Other (weighted) identities on $L^2(\mathbb{G})$

In [R.-Suragan, AM, 2017] and [R.-Suragan-Yessirkegenov, IEOT, 2018] we extended such ideas to further settings: e.g. For all $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, for every $f \in C_0^\infty(\mathbb{G}\backslash\{0\}), \alpha \in \mathbb{R}$, and any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$ we have:

$$\left\|\frac{1}{|x|^\alpha}\mathbb{E}f\right\|^2_{L^2(\mathbb{G})} = \left(\frac{Q}{2}-\alpha\right)^2\left\|\frac{f}{|x|^\alpha}\right\|^2_{L^2(\mathbb{G})} + \left\|\frac{1}{|x|^\alpha}\mathbb{E}f + \frac{Q-2\alpha}{2|x|^\alpha}f\right\|^2_{L^2(\mathbb{G})}.$$

$$\left\|\frac{1}{|x|^\alpha}\mathcal{R}^k_{|x|}f\right\|^2_{L^2(\mathbb{G})} = \left[\prod_{j=0}^{k-1}\left(\frac{Q-2}{2}-(\alpha+j)\right)^2\right]\left\|\frac{f}{|x|^{k+\alpha}}\right\|^2_{L^2(\mathbb{G})}$$

$$+\sum_{l=1}^{k-1}\left[\prod_{j=0}^{l-1}\left(\frac{Q-2}{2}-(\alpha+j)\right)^2\right]\left\|\frac{1}{|x|^{l+\alpha}}\mathcal{R}^{k-l}_{|x|}f + \frac{Q-2(l+1+\alpha)}{2|x|^{l+1+\alpha}}\mathcal{R}^{k-l-1}_{|x|}f\right\|^2_{L^2(\mathbb{G})}$$

$$+\left\|\frac{1}{|x|^\alpha}\mathcal{R}^k_{|x|}f + \frac{Q-2-2\alpha}{2|x|^{1+\alpha}}\mathcal{R}^{k-1}_{|x|}f\right\|^2_{L^2(\mathbb{G})}.$$

Rellich identity

Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 5$. Let $|\cdot|$ be any homogeneous quasi-norm on $\mathbb{G}$. Then for every $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$:

$$\left\| \mathcal{R}^2 f + \frac{Q-1}{|x|}\mathcal{R}f + \frac{Q(Q-4)}{4|x|^2} f \right\|^2_{L^2(\mathbb{G})} + \frac{Q(Q-4)}{2} \left\| \frac{1}{|x|}\mathcal{R}f + \frac{Q-4}{2|x|^2} f \right\|^2_{L^2(\mathbb{G})}$$
$$= \left\| \mathcal{R}^2 f + \frac{Q-1}{|x|}\mathcal{R}f \right\|^2_{L^2(\mathbb{G})} - \left(\frac{Q(Q-4)}{4} \right)^2 \left\| \frac{f}{|x|^2} \right\|^2_{L^2(\mathbb{G})}.$$

Corollary: Rellich inequality

$$\left\| \frac{f}{|x|^2} \right\|_{L^2(\mathbb{G})} \leq \frac{4}{Q(Q-4)} \left\| \mathcal{R}^2 f + \frac{Q-1}{|x|}\mathcal{R}f \right\|_{L^2(\mathbb{G})}, \quad Q \geq 5.$$

The constant $\frac{4}{Q(Q-4)}$ is sharp and it is attained if and only if $f = 0$.

Weighted L^p-Hardy inequalities

Theorem

Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $\alpha \in \mathbb{R}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$, $1 < p < \infty$, and any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$ for $\alpha p \neq Q$ we have

$$\left\| \frac{f}{|x|^\alpha} \right\|_{L^p(\mathbb{G})} \leq \left| \frac{p}{Q - \alpha p} \right| \left\| \frac{1}{|x|^\alpha} \mathbb{E} f \right\|_{L^p(\mathbb{G})}.$$

If $\alpha p \neq Q$ then the constant $\left|\frac{p}{Q-\alpha p}\right|$ is sharp.
For $\alpha p = Q$ we have the critical case (believe to be already new in $\mathbb{R}^n$?!)

$$\left\| \frac{f}{|x|^{\frac{Q}{p}}} \right\|_{L^p(\mathbb{G})} \leq p \left\| \frac{\log|x|}{|x|^{\frac{Q}{p}}} \mathbb{E} f \right\|_{L^p(\mathbb{G})},$$

where the constant p is sharp.

Going to fractional orders

Lemma

The operator $\mathbb{A} = \mathbb{E}\mathbb{E}^*$ *is Komatsu-non-negative in* $L^2(\mathbb{G})$*:*

$$\|(\lambda + \mathbb{A})^{-1}\|_{L^2(\mathbb{G}) \to L^2(\mathbb{G})} \leq \lambda^{-1}, \forall \lambda > 0. \tag{3}$$

Since $\mathbb{A}$ is Komatsu-non-negative, we can define fractional powers of the operator $\mathbb{A}$ as in [MCSA01] and we denote

$$|\mathbb{E}|^{\beta} = \mathbb{A}^{\frac{\beta}{2}}, \qquad \beta \in \mathbb{C}.$$

C. C. Martinez and M. A. Sanz.
The theory of fractional powers of operators,
North-Holland Mathematics Studies. North-Holland Publishing Co., Amsterdam, 2001.

Example: Hardy ineq with fractional Euler operator

Theorem

Let $\mathbb{G}$ *be a homogeneous group of homogeneous dimension* Q, $\beta \in \mathbb{C}_+$ *and let* $k > \frac{\mathrm{Re}\beta}{2}$ *be a positive integer. Then for all complex-valued functions* $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ *we have*

$$\|f\|_{L^2(\mathbb{G})} \leq C(k - \frac{\beta}{2}, k) \left(\frac{2}{Q}\right)^{\mathrm{Re}\beta} \left\| |\mathbb{E}|^\beta f \right\|_{L^2(\mathbb{G})}, \; Q \geq 1, \tag{4}$$

where

$$C(\beta, k) = \frac{\Gamma(k+1)}{|\Gamma(\beta)\Gamma(k-\beta)|} \frac{2^{k-\mathrm{Re}\beta}}{\mathrm{Re}\beta(k-\mathrm{Re}\beta)}. \tag{5}$$

M. Ruzhansky, D. Suragan and N. Yessirkegenov.
Hardy-Littlewood, Bessel-Riesz, and fractional integral operators in anisotropic Morrey and Campanato spaces,
Fract. Calc. Appl. Anal., 21 (2018), 577-612.

Our new book

M. Ruzhansky and D. Suragan.
Hardy inequalities on homogeneous groups: 100 years of Hardy innequalities
Progress in Math., Vol. 327, Birkhäuser, 2019. $588pp$

• Hardy inequalities on homogeneous groups: weighted, super-weights, critical, two-weight, remainders, stability; fractional p-Laplacians;
• Rellich, Caffarelli-Kohn-Nirenberg, Sobolev type: stability, weights, higer orders
• Horizontal inequalities on stratified groups: Hardy-Rellich, higher orders, two-weight, factorisations, drifts, p-sub-Laplacians
• Hardy-Rellich and fundamental solutions: drifts, complex affine groups, Baouendi-Grushin, CKN
• Uncertainty relations on homogeneous groups: position-momentum relations, Heisenberg-Kennard, Pythagorean, Euler-Coulomb, Heisenberg-Pauli-Weyl
• Function spaces on homogeneous groups: Euler-Hilbert-Sobolev, Sobolev-Lorentz-Zygmund, Morrey, Campanato, Poincaré inequalities, Hardy-Littlewood maximal function, Bessel-Riesz, Olsen type inequalities
• Potential theory on stratified groups: layer potentials, BVPs, Kac problem, Newton potentials, Green functions, p-sub-Laplacian
• Sums of squares: local Hardy, Rellich, uncertainty, Green's identities, perimeter and surface measures

Come back to hypoelliptic functional inequalities

Let $\mathbb{G}$ be a stratified group of homogeneous dimension Q, with homogeneous norm $|\cdot|$. Let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$. Then we have

$$\left\| \frac{f}{|x|^{\alpha}} \right\|_{L^p(\mathbb{G})} \leq C \|(-\mathcal{L})^{\alpha/2} f\|_{L^p(\mathbb{G})}, \; 1 < p < \infty, \; 0 < \alpha < Q/p.$$

The proof is based on the positiveness of the integral kernel of the operator $T_\alpha f = |\cdot|^{-\alpha}(-\mathcal{L})^{-\alpha/2} f = |\cdot|^{-\alpha}(f * I_\alpha)$ (where I_α is the Riesz potential of order α), which we do not have for higher order operators.

P. Ciatti, M. Cowling and F. Ricci.
Hardy and uncertainty inequalities on stratified Lie groups.
Adv. Math., 277:365–387, 2015.

Questions: what about other hypoelliptic operators on the right hand side? not just sub-Laplacian, not second order;
what is the nilpotent Lie group is not stratified?

Answer: not much has been known (before)!

Positive Rockland operators

Recall: Rockland operator $\mathcal{R}$ is a homogeneous hypoelliptic invariant differential operator on a nilpotent Lie group. (after Helffer and Nourrigat)
To give some examples, this setting includes:

- for $\mathbb{G} = \mathbb{R}^n$, $\mathcal{R}$ may be any positive homogeneous elliptic differential operator with constant coefficients. For example, we can take

$$\mathcal{R} = (-\triangle)^m \text{ or } \mathcal{R} = (-1)^m \sum_{j=1}^{n} a_j \left(\frac{\partial}{\partial x_j}\right)^{2m}, \; a_j > 0, \; m \in \mathbb{N};$$

- for $\mathbb{G} = \mathbb{H}_n$ the Heisenberg group, we can take

$$\mathcal{R} = (-\mathcal{L})^m \text{ or } \mathcal{R} = (-1)^m \sum_{j=1}^{n} (a_j X_j^{2m} + b_j Y_j^{2m}), a_j, b_j > 0, m \in \mathbb{N},$$

where $\mathcal{L}$ is the sub-Laplacian and X_j, Y_j are the left invariant vector fields.

- for any stratified Lie group (or homogeneous Carnot group) with vectors $X_1, \ldots, X_k$ spanning the first stratum, we can take

$$\mathcal{R} = (-1)^m \sum_{j=1}^{k} a_j X_j^{2m}, \ a_j > 0,$$

so that in particular, for $m = 1$, $\mathcal{R}$ is a positive sub-Laplacian;
- for any graded Lie group $\mathbb{G} \sim \mathbb{R}^n$ with dilation weights $\nu_1, \ldots, \nu_n$ let us fix the basis $X_1, \ldots, X_n$ of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ satisfying

$$D_r X_j = r^{\nu_j} X_j, \ j = 1, \ldots, n, r > 0,$$

where D_r denote dilations on the Lie algebra. If ν_0 is any common multiple of $\nu_1, \ldots, \nu_n$, the operator

$$\mathcal{R} = \sum_{j=1}^{n} (-1)^{\frac{\nu_0}{\nu_j}} a_j X_j^{2\frac{\nu_0}{\nu_j}}, a_j > 0$$

is a Rockland operator of homogeneous degree $2\nu_0$.

Some discussion of Rockland operators

- Rockland operator $\mathcal{R}$ is a homogeneous hypoelliptic invariant differential operator on a nilpotent Lie group.
- $\mathcal{R}$ exists $\Leftrightarrow$ dilation weights are rational $\Leftrightarrow$ the group is graded.
- Folland and Stein: this is a natural setting to combine harmonic analysis & PDEs. (and the usual working assumption)
- Rockland condition: for every representation $\pi \in \widehat{\mathbb{G}}$, except for the trivial representation, the operator $\pi(\mathcal{R})$ is injective on $\mathcal{H}_\pi^\infty$, that is,

$$\forall v \in \mathcal{H}_\pi^\infty, \quad \pi(\mathcal{R})v = 0 \Rightarrow v = 0.$$

  Here $\pi(\mathcal{R}) := d\pi(\mathcal{R})$ is the infinitesimal representation of the Rockland operator $\mathcal{R}$ as of an element of the universal enveloping algebra of $\mathbb{G}$.
- Rockland, R. Beals, Helffer-Nourrigat (1979): equivalence of above conditions
- Fischer-R. (Ann. Inst. Fourier 2017, Birkhäuser book 2016): Sobolev spaces, Fourier and spectral multipliers, pseudo-differential operators
- Cardona-R. (CRAS 2017) Besov spaces, Fourier and spectral multipliers on Besov spaces

Hardy-Sobolev inequality on graded groups

Hardy-Sobolev inequality (R.+Yessirkegenov, 2018)

Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Let $|\cdot|$ be an arbitrary homogeneous quasi-norm. Let $1 < p \leq q < \infty$ and $0 < a < Q/p$. Let $0 \leq b < Q$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Then $\exists C > 0$:

$$\left\| \frac{f}{|x|^{\frac{b}{q}}} \right\|_{L^q(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})}$$

- if $\mathbb{G}$ is stratified, $\mathcal{R} = -\mathcal{L}$, $p = q \longrightarrow$ Ciatti, Cowling and Ricci 2015.
- if $b = 0 \longrightarrow$ Sobolev embedding (Fischer+R.) $1 < p < q < \infty$, $0 < a < Q/p$, $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q}$. Then $\|f\|_{L^q(\mathbb{G})} \leq C\|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})}$.
- $p = q$ and $a = 1 \longrightarrow$ Hardy $\left\| \frac{f}{|x|} \right\|_{L^p(\mathbb{G})} \leq C\|\mathcal{R}^{\frac{1}{\nu}} f\|_{L^p(\mathbb{G})}$, $1 < p < Q$.
- if $p = q$ and $a = 2 \longrightarrow$ Rellich $\left\| \frac{f}{|x|^2} \right\|_{L^p(\mathbb{G})} \leq C\|\mathcal{R}^{\frac{2}{\nu}} f\|_{L^p(\mathbb{G})}$, $1 < p < Q/2$.

Critical Hardy-Sobolev inequalities on graded groups

Critical global Hardy-Sobolev inequality $a = Q/p$

Let $1 < p < r < \infty$ and $p < q < (r-1)p'$, $1/p + 1/p' = 1$. Then $\exists C = C(p, q, r, Q) > 0$: for all $f \in L^p_{Q/p}(\mathbb{G})$ we have

$$\left\| \frac{f}{\left(\log\left(e + \frac{1}{|x|}\right)\right)^{\frac{r}{q}} |x|^{\frac{Q}{q}}} \right\|_{L^q(\mathbb{G})} \leq C \|f\|_{L^p_{Q/p}(\mathbb{G})}$$

Critical local Hardy-Sobolev inequality $a = Q/p$

Let $1 < p < \infty$ and $\beta \in [0, Q)$. Let $r > 0$ and let x_0 be any point of $\mathbb{G}$. Then for any $p \leq q < \infty$ there exists $C_4 = C_4(p, Q, \beta, r, q)$ such that

$$\left\| \frac{f}{|x|^{\frac{\beta}{q}}} \right\|_{L^q(B(x_0,r))} \leq C_4 q^{1-1/p} \|f\|_{L^p_{Q/p}(B(x_0,r))}$$

and $\limsup_{q\to\infty} C_4(p, Q, \beta, r, q) < \infty$.

Trudinger-Moser inequalities on graded groups

Interesting: Local Hardy $\Longleftrightarrow$ local Trudinger-Moser, with asymptotic relation between best constants C_4 (Hardy) and C_2 (local TM)!

Local Trudinger-Moser inequality (with remainder terms)

Let $1 < p < \infty$ and $\beta \in [0, Q)$. Let $r > 0$ and let x_0 be any point of $\mathbb{G}$. Then

$$\int_{B(x_0,r)} \frac{1}{|x|^\beta} \left(\exp(\alpha |f(x)|^{p'}) - \sum_{0 \leq k < p-1} \frac{\alpha^k |f(x)|^{kp'}}{k!} \right) dx \leq C_1 \|f\|^p_{L^p_{Q/p}(B(x_0,r))},$$

for any $\alpha \in [0, C_2)$ and for all $f \in L^p_{Q/p}(B(x_0, r))$ with $\|f\|_{L^p_{Q/p}(B(x_0,r))} \leq 1$.

Global Trudinger-Moser inequality (with remainder terms)

Let $1 < p < \infty$, $\beta \in [0, Q)$, $\mu > Q/(Q - \beta)$. Then for all $\alpha \in (0, C_2)$ and $\|\mathcal{R}^{\frac{Q}{\nu p}} f\|_{L^p(G)} \leq 1$:

$$\int_{\mathbb{G}} \frac{1}{|x|^\beta} \left(\exp(\alpha |f(x)|^{p'}) - \sum_{0 \leq k < p-1} \frac{\alpha^k |f(x)|^{kp'}}{k!} \right) dx \leq C_3 (\|f\|^p_{L^p(G)} + \|f\|^{p/\mu}_{L^p(\mathbb{G})}).$$

Gagliardo-Nirenberg inequality on graded groups

Interesting: Global Trudinger-Moser $\Longleftrightarrow$ weighted Gagliardo-Nirenberg, with asymptotic relation between best constants C_2 (TM) and C_7 (GN)!

Weighted Gagliardo-Nirenberg on graded groups

Let $1 < p < \infty$, $\beta \in [0, Q)$, $Q/(Q-\beta) < \mu < \infty$. Then for any $p \leq q < \infty$ there exists $C_7 = C_7(p, Q, \beta, \mu, q) > 0$ such that

$$\left\| \frac{f}{|x|^{\frac{\beta}{q}}} \right\|_{L^q(\mathbb{G})} \leq C_7 q^{1-1/p} \left(\|\mathcal{R}^{\frac{Q}{\nu p}} f\|_{L^p(\mathbb{G})}^{1-p/q} \|f\|_{L^p(\mathbb{G})}^{p/q} + \|\mathcal{R}^{\frac{Q}{\nu p}} f\|_{L^p(\mathbb{G})}^{1-p/(q\mu)} \|f\|_{L^p(\mathbb{G})}^{p/(q\mu)} \right)$$

and such that $\limsup\limits_{q\to\infty} C_7(p, Q, \beta, \mu, q) < \infty$.

☞ In the case $\beta = 0$ this also yields that $L^p_{Q/p}(\mathbb{G})$ is continuously embedded in $L^q(\mathbb{G})$ for any $1 < p < \infty$, $p \leq q < \infty$.

☞ For $\mathbb{G} = (\mathbb{R}^n, +)$ (Ozawa'95), for Heisenberg group and $Q/p = 1$ (Yang'14).

☞ For graded groups, this is the critical case of $L^p_a(\mathbb{G}) \hookrightarrow L^q(\mathbb{G})$ with $1/q = 1/p - a/Q$ and $0 < a < Q/p$ (Fischer+R., AIF 2016).

Caffarelli-Kohn-Nirenberg inequality on graded groups

Caffarelli-Kohn-Nirenberg on graded groups (R.+Yessirkegenov 2018)

Let $1 < p, q < \infty$, $\delta \in (0,1]$ and $0 < r < \infty$ with $r \le \frac{q}{1-\delta}$ for $\delta \neq 1$. Let $0 < a < Q/p$ and β, $\gamma \in \mathbb{R}$ with $\delta r(Q - ap - \beta p) \le p(Q + r\gamma - r\beta)$ and $\beta(1-\delta) - \delta a \le \gamma \le \beta(1-\delta)$. Assume that $\frac{r(\delta Q + p(\beta(1-\delta) - \gamma - a\delta))}{pQ} + \frac{(1-\delta)r}{q} = 1$. Then there exists $C > 0$ such that for all $f \in \dot{L}^p_a(\mathbb{G})$ we have

$$\left\| |x|^{\gamma} f \right\|_{L^r(\mathbb{G})} \le C \left\| \mathcal{R}^{\frac{a}{\nu}} f \right\|^{\delta}_{L^p(\mathbb{G})} \left\| |x|^{\beta} f \right\|^{1-\delta}_{L^q(\mathbb{G})}$$

☞ This extends the range of indices compared to the classical CKN inequality.

☞ for $\beta = \gamma = 0$ this gives the Garliardo-Nirenberg inequality $\|f\|_{L^r(\mathbb{G})} \le C \left\| \mathcal{R}^{\frac{a}{\nu}} f \right\|^{\delta}_{L^p(\mathbb{G})} \|f\|^{1-\delta}_{L^q(\mathbb{G})}$ obtained by R.+Tokmagambetov (2016) with applications to global-in-time well-pos. of nonlinear hypoelliptic wave equations.

☞ There are analogous CKN with Euler operators on homogeneous groups:

M. Ruzhansky, D. Suragan and N. Yessirkegenov.
Extended Caffarelli-Kohn-Nirenberg inequalities, and remainders, stability, and superweights for L_p-weighted Hardy inequalities. *Trans. AMS. Ser. B*, 2018.

Hardy-Littlewood-Sobolev inequalities

Hardy-Littlewood-Sobolev inequalities on homogeneous groups

Let $0 < \lambda < Q$ and $1 < p, q < \infty$ be such that $1/p + 1/q + (\alpha + \lambda)/Q = 2$ with $0 \leq \alpha < Q/p'$ and $\alpha + \lambda \leq Q$. Then there exists $C(Q, \lambda, p, \alpha) > 0$:

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)} g(y)}{|x|^{\alpha} |y^{-1}x|^{\lambda}} dx dy \right| \leq C \|f\|_{L^p(\mathbb{G})} \|g\|_{L^q(\mathbb{G})}.$$

Hardy-Littlewood-Sobolev inequalities on graded groups

Let $1 < p, q < \infty$, $0 \leq a < Q/p$ and $0 \leq b < Q/q$. Let $0 < \lambda < Q$, $0 \leq \alpha < a + Q/p'$ and $0 \leq \beta \leq b$ be such that $\alpha + \lambda \leq Q$, $(Q - ap)/(pQ) + (Q - q(b - \beta))/(qQ) + (\alpha + \lambda)/Q = 2$. Then $\exists C > 0$:

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)} g(y)}{|x|^{\alpha} |y^{-1}x|^{\lambda} |y|^{\beta}} dx dy \right| \leq C \|f\|_{\dot{L}^p_a(\mathbb{G})} \|g\|_{\dot{L}^q_b(\mathbb{G})}.$$

- ☞ $a = b = 0$: Heisenberg group (Folland-Stein), best constant (Lieb-Frank);
- ☞ some results on reverse HLS; $\mathbb{R}^n$: Carrillo-Delgadino-Patacchini, Dolbeault-Frank-Hoffmann (2018); groups: Kassymov+Suragan+R. (2019)

Some ideas of proof

Some parts of proofs are based on the following integral Hardy inequalities on homogeneous groups, by working with Riesz kernels of positive Rockland operators. Let $1 < p \leq q < \infty$ and $\{\phi_i\}_{i=1}^2, \{\psi_i\}_{i=1}^2 \geq 0$. Then the inequalities

$$\left(\int_{\mathbb{G}}\left(\int_{B(0,|x|)} f(z)dz\right)^q \phi_1(x)dx\right)^{\frac{1}{q}} \leq C_1 \left(\int_{\mathbb{G}}(f(x))^p\psi_1(x)dx\right)^{\frac{1}{p}}$$

and $\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,|x|)} f(z)dz\right)^q \phi_2(x)dx\right)^{\frac{1}{q}} \leq C_2 \left(\int_{\mathbb{G}}(f(x))^p\psi_2(x)dx\right)^{\frac{1}{p}}$
hold for all $f \geq 0$ a.e. on $\mathbb{G}$ if and only if, respectively, we have

$$A_1 := \sup_{R>0}\left(\int_{\{|x|\geq R\}} \phi_1(x)dx\right)^{\frac{1}{q}}\left(\int_{\{|x|\leq R\}}(\psi_1(x))^{-(p'-1)}dx\right)^{\frac{1}{p'}} < \infty$$

and $A_2 := \sup_{R>0}\left(\int_{\{|x|\leq R\}} \phi_2(x)dx\right)^{\frac{1}{q}}\left(\int_{\{|x|\geq R\}}(\psi_2(x))^{-(p'-1)}dx\right)^{\frac{1}{p'}} < \infty.$

- ☞ We also have a similar characterisation for $p > q$.
- ☞ Holds on metric measure spaces (R.+Verma, Proc. Royal Soc. A., 2019).
- ☞ Symmetric spaces and Damek-Ricci spaces (Anker+Kassymov+R., 2020).

Further remarks

☞ **(Gagliardo-Nirenberg inequality)** (R.+Tokmagambetov, JDE, 2018)
For the study of

$$\partial_t^2 u - \mathcal{R}u + b\partial_t u + mu = f(u, \{\mathcal{R}^{j/\nu}\}_{j=1}^{[\nu/2]-1}),\ u(0) \in H^{\nu/2},\ \partial_t u(0) \in L^2:$$

Let $\mathbb{G}$ be graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Let $a > 0$, $1 < p < \frac{Q}{a}$ and $p \leq q \leq \frac{pQ}{Q-ap}$. Then

$$\int_{\mathbb{G}} |u(x)|^q dx \leq C \left(\int_{\mathbb{G}} |\mathcal{R}^{\frac{a}{\nu}} u(x)|^p dx \right)^{\frac{Q(q-p)}{ap^2}} \left(\int_{\mathbb{G}} |u(x)|^p dx \right)^{\frac{apq-Q(q-p)}{ap^2}}.$$

☞ **(Sobolev inequality)**

$$\left(\int_{\mathbb{G}} |u(x)|^q dx \right)^{\frac{p}{q}} \leq C \int_{\mathbb{G}} (|\mathcal{R}^{\frac{a}{\nu}} u(x)|^p + |u(x)|^p) dx.$$

☞ (R.+Tokmagambetov+Yessirkegenov 2016) Best constants can related to ground states (least energy solutions) of the following Schrödinger-type equation:

$$\mathcal{R}^{\frac{a}{\nu}} (|\mathcal{R}^{\frac{a}{\nu}} u(x)|^{p-2} \mathcal{R}^{\frac{a}{\nu}} u(x)) + |u(x)|^{p-2} u(x) = |u(x)|^{q-2} u(x).$$

☞ This is also a differential equation if $\frac{a}{\nu}$ is an integer.

Further remarks

☞ They are also related to the variational problem

$$d = \inf_{\substack{u \in L^p_a(\mathbb{G}) \setminus \{0\} \\ \mathfrak{I}(u)=0}} \mathfrak{L}(u),$$

for functionals

$$\mathfrak{L}(u) = \frac{1}{p} \int_{\mathbb{G}} |\mathcal{R}^{\frac{a}{\nu}} u(x)|^p dx + \frac{1}{p} \int_{\mathbb{G}} |u(x)|^p dx - \frac{1}{q} \int_{\mathbb{G}} |u(x)|^q dx$$

and

$$\mathfrak{I}(u) = \int_{\mathbb{G}} (|\mathcal{R}^{\frac{a}{\nu}} u(x)|^p + |u(x)|^p - |u(x)|^q) dx.$$

☞ Other results: critical case of $\phi \in L^p_{Q/p}(\mathbb{G})$, and its links to variational problems are related to hypoelliptic versions of critical Gagliardo-Nirenberg, Trudinger-Moser and Brezis-Gallouet-Wainger inequalities.

☞ Linear and nonlinear evolution PDEs for hypoelliptic operators (e.g. heat, wave, Schrödinger; very weak solutions)

☞ Many further open questions and problems ....

Fourier and spectral multipliers

Spectral multipliers on graded groups, Rottensteiner+R., 2019

Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν on a graded group G of homogeneous dimension Q. Let φ be a monotonically decreasing function on $[0,\infty)$ such that $\varphi(0)=1$, $\lim_{\lambda\to\infty}\varphi(\lambda)=0$. Then

$$\|\varphi(\mathcal{R})\|_{L^p(G)\to L^q(G)} \lesssim \sup_{\lambda>0}\varphi(\lambda)\lambda^{\frac{Q}{\nu}(\frac{1}{p}-\frac{1}{q})}, \quad 1<p\leq 2\leq q<\infty.$$

☞ Application: $\|e^{-t\mathcal{R}}\|_{L^p\to L^q}\lesssim t^{-\frac{Q}{\nu}(\frac{1}{p}-\frac{1}{q})}$ as $t\to\infty$.
☞ Sharpness: this yields Sobolev embedding $L^p_{s_1}(G)\subset L^q_{s_2}(G)$ for $s_1-s_2=\frac{Q}{\nu}(\frac{1}{p}-\frac{1}{q})$. (known from Fischer+R., AIF 2017)
☞ $\mathbb{R}^n$: follows from Hörmander, 1960 AM paper.
☞ Compact Lie groups (sublaplacians): Akylzhanov+R., JFA 2019, Q – Hausdorff dimension. Also for the Heisenberg group, and for general unimodular locally compact groups (which is also partly used in this proof)
☞ Compact quantum groups (spectral triples, A.+Majid+R., CMP 2018)
☞ Damek-Ricci spaces (harmonic NA groups): Anker+Kumar+R., 2020

Locally compact groups (Akylzhanov+R., JFA 2019)

Crucial observation: can think of Hörmander's result on $\mathbb{R}^n$, $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, as

$$\|A\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \lesssim \sup_{s>0} s \left(\int\limits_{\substack{\xi\in\mathbb{R}^n \\ |\sigma_A(\xi)|\geq s}} d\xi \right)^{\frac{1}{r}} \simeq \|\sigma_A\|_{L^{r,\infty}(\mathbb{R}^n)} \simeq \|A\|_{L^{r,\infty}(VN(\mathbb{R}^n))}.$$

Fourier multipliers on locally compact groups

Let G be a locally compact unimodular group, and let A be a linear continuous operator on the Schwartz-Bruhat space $\mathcal{S}(G)$ (affiliated with $VN_R(G)$). Let $1 < p \leq 2 \leq q < +\infty$, and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Then we have

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \|A\|_{L^{r,\infty}(VN_R(G))} \simeq \sup_{s>0} s \left[\int\limits_{t\in\mathbb{R}_+ :\ \mu_t(A)\geq s} dt \right]^{\frac{1}{p}-\frac{1}{q}}.$$

The distribution function $d_\lambda(A) = \tau(E_{(\lambda,+\infty)}(|A|))$ and (Thierry+Kosaki) $\mu_t(A) := \inf\{\lambda \geq 0 \colon d_\lambda(A) \leq t\}$ are generalised t-th singular numbers.

Spectral multipliers (Akylzhanov+R., JFA 2019)

Corollary: spectral multipliers

Let G be a locally compact separable unimodular group and let A be a left Fourier multiplier on G (i.e. affiliated with $VN_R(G)$). Let φ be monotonic on $[0,+\infty)$ such that $\varphi(0)=1$, $\lim_{u\to+\infty}\varphi(u)=0$. Then

$$\|\varphi(|A|)\|_{L^p(G)\to L^q(G)} \lesssim \sup_{u>0}\varphi(u)\left[\tau(E_{(0,u)}(|A|))\right]^{\frac{1}{p}-\frac{1}{q}}, \quad 1<p\leq 2\leq q<\infty.$$

Here τ is the canonical (semifinite faithful trace) on $VN_R(G)$, and $E_{(0,u)}(|A|)$ – spectral projection of $|A|$ corresponding on $(0,u)$.

☞ a version of this result holds also for non-invariant operators;

☞ For any just Borel measurable φ on $Sp(|A|)$ it holds $\|\varphi(|A|)\|_{L^p(G)\to L^q(G)} \lesssim \sup_{s>0} s[\tau(E_{(s,+\infty)}(\varphi(|A|)))]^{\frac{1}{p}-\frac{1}{q}}$.

☞ Spectral multipliers result on graded groups follows from this and $\tau(E_{(0,u)}(\mathcal{R}))\sim u^{\frac{Q}{\nu}}$. This can be deduced by Kirilov's orbit method from Ter Elst-Robinson results on spectra of $\pi(\mathcal{R})$ for $\pi\in\widehat{G}$.

Thank you