Global attraction to solitary waves

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Abstract

We prove that any finite energy solution to the Klein–Gordon equation with a nonlinearity localized on a compact part of a hyperplane (a) has a compact spectrum; (b) is a solitary wave.

We also prove that finite energy solutions to the nonlinear Schrödinger equation and nonlinear Klein–Gordon equation which have the compact spectrum have to be solitary waves.

Soliton resolution conjecture:

In a nonlinear dispersive system,

long-time asymptotics of any finite energy solution

is a superposition of outgoing solitary waves

and a dispersive wave

[Komech^{03}, Soffer^{06}, Tao^{07}, Duyckaerts et al.^{16}]...

One of the strategies [Komech^{03}]**:**

1. Any solution converges to radiationless solution with compact spectrum;
2. A solution with compact spectrum in fact has a spectrum consisting of one frequency: $u(t,x) = \phi(x)e^{-\mathrm{i}\omega t}$, $\phi \in H^1(\mathbb{R}^n, \mathbb{C})$.

based on the **Titchmarsh convolution theorem**:

Let $f,\, g \in L^1(\mathbb{R})$; $\operatorname{supp} f$, $\operatorname{supp} g$ are compact

Note: $\operatorname{supp}(f * g) \subset \operatorname{supp} f + \operatorname{supp} g$

[Titchmarsh[26]]:
$$\boldsymbol{\inf \operatorname{supp} f * g = \inf \operatorname{supp} f + \inf \operatorname{supp} g,}$$
$$\boldsymbol{\sup \operatorname{supp} f * g = \sup \operatorname{supp} f + \sup \operatorname{supp} g.}$$

Example: $2\cos^2 t = 1 + \cos 2t$,
$$2\,\frac{\delta_1(\omega)+\delta_{-1}(\omega)}{2} * \frac{\delta_1(\omega)+\delta_{-1}(\omega)}{2} = \delta(\omega) + \frac{\delta_2(\omega)+\delta_{-2}(\omega)}{2}$$

“Counterexample”: $\delta' * 1 = 0$

[Lions[51]]: $\mathrm{CH}\operatorname{supp} f * g = \mathrm{CH}\operatorname{supp} f + \mathrm{CH}\operatorname{supp} g \qquad \forall f, g \in \mathscr{E}'(\mathbb{R}^N).$

CH is the convex hull of a set.

Example: Klein–Gordon with concentrated nonlinearity

$$-\partial_t^2 u = -\partial_x^2 u + m^2 u - \delta(x)|u|^2 u, \qquad u(t,x) \in \mathbb{C}, \qquad x \in \mathbb{R}$$

[Komech[03], Komech & Komech[07]]:

A weak global attractor is formed by the set of solitary waves.

Equivalently, **an Ω-limit of a finite energy solution is a solitary wave.**

Proof: Denote RHS by $f(t) := |u(t,0)|^2 u(t,0)$

- $\tilde{f}(\omega)$ is L^2_{loc} on $\mathbb{R} \setminus [-m,m]$;
- Ω-lim point: $(v_0, v_1) = \lim\limits_{T_j \to +\infty} (u, \partial_t u)|_{t=T_j}$ in $H^1_{\mathrm{loc}}(\mathbb{R}) \times L^2_{\mathrm{loc}}(\mathbb{R})$;
- Ω-lim trajectory: $v(t,x)$ with $(v, \partial_t v)|_{t=0} = (v_0, v_1)$;
- $g(t) := |v(t,0)|^2 v(t,0)$ satisfies $\tilde{g}(\omega) \equiv 0$ on $\mathbb{R} \setminus [-m,m]$;
- Let $V(t) := v(t,0)$; $\operatorname{supp} \tilde{V}(\omega) \subset [-m,m]$, $2\sqrt{m^2-\omega^2}\,\tilde{V}(\omega) = \tilde{V} * \tilde{\bar{V}} * \tilde{V}$;
- The Titchmarsh theorem: $\operatorname{supp} \tilde{V}(\omega) = \{\omega_0\}$, $\tilde{v}(\omega,x) = \delta(\omega-\omega_0)\phi(x)$.

“Counterexample”: breathers in integrable systems

sine-Gordon equation:

$$-\partial_t^2 u = -\partial_x^2 u + \sin u, \qquad u(t,x) \in \mathbb{R}, \quad x \in \mathbb{R}$$

Exact solution [Ablowitz et al.[73]]:

$$u(t,x) = 4 \arctan\left(\frac{\sqrt{1-\omega^2}\cos(\omega t)}{\omega \cosh(\sqrt{1-\omega^2}\, x)} \right), \qquad \omega \in (-1,1)$$

• Spectrum is not compact; • the nonlinearity is not of algebraic type.

The cubic nonlinear Schrödinger equation:

$$\mathrm{i}\partial_t u = -\partial_x^2 u - 2|u|^2 u, \qquad u(t,x) \in \mathbb{C}, \quad x \in \mathbb{R}$$

Exact solution [Akhmediev et al.[87]]:

$$u(t,x) = \frac{\cos x + \mathrm{i}\sqrt{2}\sinh t}{\sqrt{2}\cosh t - \cos x} e^{\mathrm{i}t}$$

• Spectrum is not compact; • infinite L^2-norm.

Multifrequency solitary waves in KG

[Komech & Komech[09]]: KG with mean field self-interaction,

$$E = \frac{1}{2}\int_{\mathbb{R}^n} \left(|\partial_t u|^2 + |\nabla u|^2 + m^2|u|^2\right)dx \quad + \quad |\langle \rho, u(t,\cdot)\rangle|^4, \qquad \rho \in \mathscr{S}(\mathbb{R}^n)$$

[Komech & Komech[10]]: KG with nonlinear oscillators $\sim |u|^2 u$ at $x = \pm R$,

$$u(t,x) = \begin{cases} e^{-\kappa|x|}\sin t, & |x| > R; \\ a\cosh(\kappa|x|)\sin t + b\cos(K|x|)\sin 3t, & |x| \le R. \end{cases}$$

[Barashenkov et al.[12]]: breathers in NLS systems, $\phi_1(x)e^{-\mathrm{i}\omega_1 t} + \phi_2(x)e^{-\mathrm{i}\omega_2 t}$

Multifrequency solitary waves in KG on $\mathbb{Z}\times\mathbb{Z}$

NLKG in discrete space-time:

$$-D_T^2 u_X^T = -D_X^2 u_X^T + m^2\,\frac{u_X^{T+1}+u_X^{T-1}}{2} - \frac{W\big(|u_0^{T+1}|^2\big)-W\big(|u_0^{T-1}|^2\big)}{|u_0^{T+1}|^2-|u_0^{T-1}|^2}\big(u_0^{T+1}+u_0^{T-1}\big)$$

[Comech[13]]: 1-, 2-, and 4-frequency solitary waves:

$\phi_X e^{-\mathrm{i}\omega T}$

$\phi_X e^{-\mathrm{i}\omega T} + \psi_X e^{-\mathrm{i}(\omega+\pi)T}$

$\phi_X e^{-\mathrm{i}\omega_1 T} + \psi_X e^{-\mathrm{i}(\omega_1+\pi)T} + \alpha_X e^{-\mathrm{i}\omega_2 T} + \beta_X e^{-\mathrm{i}(\omega_2+\pi)T}$

$\phi,\ \psi\in l^2(\mathbb{Z})$;

$T\in\mathbb{Z}$ discrete time, $\quad X\in\mathbb{Z}$ discrete space.

[Comech[13]]: **Let W be polynomial, $m\delta>0$ sufficiently small. Then an Ω-lim of $u\in l^\infty(\mathbb{Z},l^2(\mathbb{Z}))$ is 1-, 2-, or 4-frequency solitary wave.**

Proof: Titchmarsh theorem for periodic distributions.

Multifrequency solitary waves in nonlinear Dirac

$$\mathrm{i}\partial_t\psi = D_m\psi - f(\psi^*\beta\psi)\beta\psi, \qquad \psi(t,x)\in\mathbb{C}^4, \qquad x\in\mathbb{R}^3.$$

[Boussaïd & Comech[18]]: bi-frequency solitary waves in NLD,

$$\phi(x)e^{-\mathrm{i}\omega t} + \psi(x)e^{\mathrm{i}\omega t} \in \mathbb{C}^4$$

Four-frequency solitary waves in NLD,

$$\phi_1(x)e^{-\mathrm{i}\omega_1 t} + \psi_1(x)e^{\mathrm{i}\omega_1 t} + \phi_2(x)e^{-\mathrm{i}\omega_2 t} + \psi_2(x)e^{\mathrm{i}\omega_2 t} \in \mathbb{C}^4.$$

Nonlinear KG, solutions with compact spectrum

$$-\partial_t^2 u = -\Delta u + m^2 u + \alpha(|u|^2)u, \qquad u(t,x) \in \mathbb{C}, \qquad x \in \mathbb{R}^n, \quad n \le 3; \quad m > 0.$$

The nonlinearity: $\alpha \in C^1(\mathbb{R}),\ \ \alpha(0) = 0.$

[Strauss[77], Berestycki & Lions[83]]: solitary wave solutions,

$$u(t,x) = \phi(x)e^{-\mathrm{i}\omega t}, \qquad \phi \in H^1(\mathbb{R}^n), \qquad \omega \in \mathbb{R}.$$

Are there multifrequency solitary waves $\phi_1(x)e^{-\mathrm{i}\omega_1 t} + \phi_2(x)e^{-\mathrm{i}\omega_2 t} + \dots$?

More generally,

Are there solutions with compact spectrum?

Spectrum of $u(t,x)$: smallest closed $\Omega \subset \mathbb{R}$ such that $\operatorname{supp}\tilde{u}(\omega,x) \subset \Omega \times \mathbb{R}^n$

Nonlinear KG, solutions with compact spectrum

Let $u \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n, \mathbb{C}))$, $x \in \mathbb{R}^n$, $n \le 3$,

$$-\partial_t^2 u = -\Delta u + m^2 u + \alpha(|u|^2)u, \qquad u(t,x) \in \mathbb{C}, \quad x \in \mathbb{R}^n,$$

with $m > 0$ and $\alpha(s)$ an algebraic function satisfying certain conditions.

Let $u \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n))$.

[Comech[19]]: **If u has compact spectrum, then it is a solitary wave:**

$$u(t,x) = \phi_0(x) e^{-\mathrm{i}\omega_0 t}, \qquad \phi_0 \in H^1(\mathbb{R}^n), \quad \omega_0 \in \mathbb{R}.$$

Proof:

- Titchmarsh theorem for partial convolution;
- UCP for $|\Delta u| \le |Vu|$, $V \in L^{n/2}_{\mathrm{loc}}(\mathbb{R}^n)$, $n \ge 2$ [Koch & Tataru[01]]

Examples of nonlinearities:

$n = 1, 2$

$$\alpha(s) = (C_0 + C_1 s + \cdots + C_p s^p)^{1/N}, \ \forall p, N \in \mathbb{N}$$

$$\alpha(s) = \frac{C_0 + \cdots + C_p s^p}{K_0 + \cdots + K_q s^q}, \quad p > q \geq 0$$

$n = 3$

$$\alpha(s) = C_0 + C_1 s + C_2 s^2$$

$$\alpha(s) = (C_0 + C_1 s)^{1/2}$$

$$\alpha(s) = \frac{C_0 + C_1 s + C_2 s^2}{K_0 + K_1 s}$$

Let $\mathrm{B} : \mathbb{R}^n \to \mathbb{R}, \quad n \in \mathbb{N}$.

Definition 1

$$\mathrm{B}^U : \mathbb{R}^n \to \mathbb{R} \text{ minimal upper semicontinuous function } \mathrm{B}^U \geq \mathrm{B};$$
$$\mathrm{B}^L : \mathbb{R}^n \to \mathbb{R} \text{ maximal lower semicontinuous function } \mathrm{B}^L \leq \mathrm{B}.$$

Let $f \in C(\mathbb{R} \times \mathbb{R}^n)$.

Definition 2 $\mathrm{B}_f(x)$ is the upper boundary of ω-support of $f(\omega, x)$:

$$\mathrm{B}_f : \mathbb{R}^n \to \mathbb{R} \sqcup \{\pm\infty\}, \quad x \mapsto \sup\left\{\omega \in \mathbb{R} \,;\, (\omega, x) \in \operatorname{supp} f\right\}; \quad \mathrm{B}_f = \mathrm{B}_f^U.$$

For $f, g \in \mathscr{E}'(\mathbb{R}, L^2(\mathbb{R}^n))$, let

$$(f \ast_\omega g)(\omega, x) = \int_{\mathbb{R}} f(\omega - \sigma, x) g(\sigma, x)\, d\sigma, \qquad (\omega, x) \in \mathbb{R} \times \mathbb{R}^n.$$

Titchmarsh theorem for partial convolution:

[Comech[19]]: **For all** $f, g \in \mathscr{E}'(\mathbb{R}, L^2_{\mathrm{loc}}(\mathbb{R}^n))$**,**

$$\mathrm{B}_{f \ast_\omega g} = \left(\mathrm{B}_f^L + \mathrm{B}_g\right)^U = \left(\mathrm{B}_f + \mathrm{B}_g^L\right)^U.$$

Klein–Gordon with nonlinearity on a hyperplane

Let $\quad u\in L^\infty(\mathbb{R},H^1(\mathbb{R}^n,\mathbb{C})),\quad x=(x',x_n)\in\mathbb{R}^n,\ n\in\mathbb{N},$

$$-\partial_t^2 u=-\Delta u+m^2u+\rho(x')\delta(x_n)\alpha(|u|^2)u,\qquad \rho\in C_{\mathrm{comp}}(\mathbb{R}^{n-1})$$

with $m>0$ and $\alpha(s)$ an algebraic function satisfying certain conditions.

[Comech[20]]: **Any Ω-lim of a finite energy solution is a solitary wave.**

Equivalently,

$$(u,\partial_t u)|_t \underset{t\to\pm\infty}{\longrightarrow} \{\phi(x),-\mathrm{i}\omega\phi(x)\}\quad\text{in }\ H^{1-\varepsilon}_{-\delta}(\mathbb{R}^n)\times H^{-\varepsilon}_{-\delta}(\mathbb{R}^n),\ \ \varepsilon,\delta>0$$

with $\phi(x)e^{-\mathrm{i}\omega t}$ solitary waves; $\quad H^s_b(\mathbb{R}^n):=\{u\ ;\ \langle x\rangle^b\langle\Delta\rangle^{s/2}u\in L^2(\mathbb{R}^n)\}.$

Examples of a nonlinearity:

$$\alpha(s)=\left(\frac{P(s)}{Q(s)}\right)^{\frac{1}{N}},\quad P,\ Q\text{ polynomials},\quad \deg P-\deg Q>N.$$

Proof: $x = (x', x_n) \in \mathbb{R}^n$, $\rho \in C_{\mathrm{comp}}(\mathbb{R}^{n-1})$

Denote RHS by $f(t,x) := \rho(x')\delta(x_n)|u(t,x)|^2 u(t,x)$.

- Prove that $\hat{\hat{f}}(\omega,\xi)|_\Gamma$ is L^2_{loc} on "mass shell"

$$\Gamma := \{(\omega,\xi)\ ;\ \omega^2 = \xi^2 + m^2\};$$

- If $v(t,x)$ is Ω-limit of $u(t,x)$,

$$(v, \partial_t v)|_{t=0} = \lim_{T_j\to\infty} (u, \partial_t u)|_{t=T_j} \quad \text{in } \ H^{1-\varepsilon}_{-\delta}(\mathbb{R}^n) \times H^{-\varepsilon}_{-\delta}(\mathbb{R}^n),$$

  then $g(t,x) := \rho(x')\delta(x_n)|v(t,x)|^2 v(t,x)$ satisfies $\hat{\hat{g}}(\omega,\xi) = 0$ on Γ;

- Prove that $\operatorname{supp} \hat{\hat{v}} \subset [-m, m] \times \mathbb{R}^n$;

- Apply the Titchmarsh theorem.

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