Limits of some Besov seminorms

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(first part joint work with Giulio Tralli)
(second part joint work with Federico Buseghin and Giulio Tralli)

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I would like to thank the organisers and in particular Valentino for the kind invitation.

2 / 34

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3 / 34

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$$[f]_{p,s} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+ps}} dxdy \right)^{1/p},$$

the classical Aronszajn-Gagliardo-Slobedetzky seminorm of f.

4 / 34

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$$\lim_{s \to (\frac{1}{2})^-} (1-2s) P_s(B) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})} P(B),$$

where I have denoted by $P(B) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ the standard perimeter of B.

5 / 34

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we see that the limiting relation for $P_s(B)$ is contained in Dávila's theorem.

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$$g^{-1} \circ g' = (z' - z, \sigma' - \sigma + \frac{1}{2} < z', Jz >),$$

where I have indicated the sympletic matrix $J = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}$. Notice that $J^2 = -I$, and that $Jz = \begin{pmatrix} y \\ -x \end{pmatrix}$. The corresponding left-invariant vector fields

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$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_\sigma, \qquad X_{n+j} = \partial_{y_j} + \frac{x_j}{2}\partial_\sigma,$$

generate the Lie algebra of $\mathbb{H}^n$, since $[X_i, X_{n+j}] = \delta_{ij}\partial_\sigma$, all other commutators being trivial.

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7 / 34

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$$P_H(E) \geq C\ |E|^{\frac{Q-1}{Q}},$$

where E is a Caccioppoli set (i.e., a measurable set with finite horizontal perimeter), and Q is the so-called homogeneous dimension of the group associated to its non-isotropic scalings.

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8 / 34

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9 / 34

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9 / 34

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A result like this has far reaching implications in the development of potential theory, for instance in extension and restriction theorems, in the solvability of the Dirichlet problem, etc.

In $\mathbb{H}^n$ consider now the horizontal Laplacian $\Delta_H = \sum_{j=1}^{2n} X_j^2$. For $0 < s < 1$ consider the fractional powers of $(-\Delta_H)^s$ defined by means of Balakrishnan's classical formula

10 / 34

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Here, $P_t = e^{-t\Delta_H}$ is the heat semigroup on $\mathbb{H}^n$ constructed by B. Gaveau in his Acta 1977 paper.

11 / 34

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In this case, we call the number $\mathfrak{P}_{H,s}(E) \in [0, \infty)$ the horizontal s-perimeter of E.

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12 / 34

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It is natural to ask whether a similar phenomenon holds in the sub-Riemannian setting of $\mathbb{H}^n$.

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13 / 34

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The proof of this result is not as direct as Dávila's proof in the Euclidean case. I will now give an idea of the main steps.

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14 / 34

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15 / 34

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Let $E \subset \mathbb{R}^n$ be a measurable set with finite perimeter, then for every $t > 0$ one has

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15 / 34

16 / 34

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17 / 34

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18 / 34

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19 / 34

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$$\mathscr{N}_{\alpha,p}^{\Delta}(f) = \left(\int_0^\infty \frac{1}{t^{\frac{\alpha p}{2}+1}} \int_{\mathbb{R}^n} P_t^{\Delta} \left(|f - f(x)|^p\right)(x) dxdt \right)^{\frac{1}{p}}.$$

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I want to present a quite surprising generalisation of this result.

In a series of papers, G. Tralli and I have recently developed some basic functional analytic aspects of a class of hypoelliptic and non-symmetric semigroups whose infinitesimal generators are the Kolmogorov-Fokker-Planck operators in $\mathbb{R}^{N+1}$ defined as follows:

21 / 34

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$$\mathscr{K} u = \mathscr{A} u - \partial_t u \overset{def}{=} \operatorname{tr}(Q\nabla^2 u) + < BX, \nabla u > -\partial_t u = 0,$$

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where the $N \times N$ matrices Q and B have real, constant coefficients, and $Q = Q^\star \geq 0$. I will assume throughout that $N \geq 2$, and indicate with X the generic point in $\mathbb{R}^N$, with (X, t) the one in $\mathbb{R}^{N+1}$.

The operators $\mathscr{K}$ and $\mathscr{A}$ were introduced by Hörmander in his celebrated 1967 hypoellipticity paper, where he showed that they are hypoelliptic if and only if the covariance matrix

$$K(t) = \frac{1}{t}\int_0^t e^{sB} Q e^{sB^\star}\, ds$$

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corresponding to the choice $N = 2n$, $Q = \begin{pmatrix} I_n & 0_n \\ 0_n & 0_n \end{pmatrix}$, and $B = \begin{pmatrix} 0_n & 0_n \\ I_n & 0_n \end{pmatrix}$.

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23 / 34

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- important aspect: the semigroup $\{P_t\}_{t>0}$ is in general non-symmetric and non-doubling!

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24 / 34

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24 / 34

This class of PDO's has been intensively studied over the past thirty years and thanks to the work of many people a lot is known about it. Nonetheless, some fundamental aspects presently remain elusive. The difficulties with these hypoelliptic operators stem from the fact that the drift term mixes the variables inextricably and this complicates the geometry considerably. This is already evident at the level of the model Kolmogorov equation and its probability transition kernel. Such intertwined geometries are reflected in the large time behaviour of Hörmander's fundamental solution of $(\star)$. This parallels in many respects the diverse situations that one encounters in the Riemannian setting when passing from positive to negative curvature. In general, the relevant volume function is not power like in t and need not be doubling.

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What drives the evolution however is the large time behavior of the volume function $V(t)$.

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26 / 34

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Note: The estimate $V(t) \geq c_1 t \implies (0,2) \subset \Sigma_\infty \implies D_\infty \geq 2$!

Table of $V(t)$, D_0 and D_∞

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Ex.	$\mathscr{K}$	$V(t)$	$\mathrm{tr}(B)$	N	D_0	D_∞
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(2)	$\Delta - < X, \nabla > -\partial_t$ Ornstein-Uhlenbeck	$\omega_N 2^{-\frac{N}{2}}(1-e^{-2t})^{\frac{N}{2}}$	$-N$	N	N	0
(3)	$\Delta_v + < v, \nabla_x > -\partial_t$ Kolmogorov	$\omega_{2n} 12^{-\frac{n}{2}} t^{2n}$	0	$2n$	$4n$	$4n$
(4)	$\partial_{vv} - x\partial_v + v\partial_x - \partial_t$ Kramers	$\pi\left(\frac{t^2}{4}+\frac{1}{8}\left(\cos(2t)-1\right)\right)^{\frac{1}{2}}$	0	2	4	2
(5)	$\partial_{vv} - 2(v+x)\partial_v + v\partial_x - \partial_t$ Smoluchowski-Kramers	$\frac{\pi}{4\sqrt{2}}\left(e^{-4t}+1-2e^{-2t}(2-\cos(2t))\right)^{\frac{1}{2}}$	-2	2	4	0
(6^+)	$\Delta_v + < v, \nabla_v > + < v, \nabla_x > -\partial_t$ Kolmogorov with friction	$\omega_{2n}\left(2e^t - \frac{t}{2} - 1 + \frac{t}{2}e^{2t} - e^{2t}\right)^n$	n	$2n$	$4n$	∞
(6^-)	$\Delta_v - < v, \nabla_v > + < v, \nabla_x > -\partial_t$ degenerate Ornstein-Uhlenbeck	$\omega_{2n}\left(2e^{-t} + \frac{t}{2} - 1 - \frac{t}{2}e^{-2t} - e^{-2t}\right)^n$	$-n$	$2n$	$4n$	$2n$

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- in Ex. 6^+ we have $D_0 < D_\infty = \infty$. $V(t) \cong t^n e^{2nt}$ is not doubling!

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$$\operatorname{tr}(B) \geq 0 \implies |P_t f(X)| \to 0 \text{ as } t \to \infty.$$

Fractional hypoelliptic operators and Sobolev spaces

29 / 34

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When $\mathscr{A} = \Delta$ these spaces coincide with the classical Aronszajn-Gagliardo-Slobedetzky spaces $W^{s,p}$!

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Therefore, it is natural to ask what is the limiting behaviour of the seminorms $\mathscr{N}_{s,p}^{\mathscr{A}}(f)$ when $s \to 0^+$.

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31 / 34

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Note that we recover (and generalise) the theorem of Maz'ya & Shaposhnikova when $\operatorname{tr} B = 0$.

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The above limit is valid both in the pointwise sense, or also in the L^p sense for any $1 < p \leq \infty$. It continues to be valid in L^1 when $\operatorname{tr} B > 0$, but it fails when $\operatorname{tr} B = 0$.

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33 / 34

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The final step of the proof is the following density result.

Proposition

For every $0 < s < 1$ and any $1 \leq p < \infty$, we have

$$\overline{\mathscr{S}}^{\mathfrak{B}^{s,p}_{\mathscr{A}}} = \mathfrak{B}^{s,p}_{\mathscr{A}}.$$

34 / 34

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Let $s \in (0, \frac{1}{2})$. Suppose that $\operatorname{tr} B \geq 0$ be valid, and that there exist $D, \gamma_D > 0$ such that $V(t) \geq \gamma_D \ t^{D/2}$ hold.

34 / 34

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$$B^{2s,1}\left(\mathbb{R}^N\right) \hookrightarrow L^{\frac{D}{D-2s}}\left(\mathbb{R}^N\right).$$

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Precisely, for every $f \in B^{2s,1}\left(\mathbb{R}^N\right)$ one has

$$||f||_{L^{\frac{D}{D-2s}}} \leq \frac{s}{i(s)\Gamma(1-s)} \mathscr{N}_{2s,1}(f),$$

where $i(s) > 0$ is the constant appearing in the nonlocal isoperimetric inequality, and $\mathscr{N}_{2s,1}(f)$ denotes the Besov seminorm.