Sobolev regularity of flows associated to non-Lipschitz vector fields

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Joint work (in progress) with L. Ambrosio[2] and S. Nicolussi Golo[3]

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Content

Flow associated to a vector field

Let us consider the Cauchy problem

$$\begin{cases} \dot\gamma(t) & = b(t,\gamma(t)) \\ \gamma(0) & = x \end{cases}, \tag{CP}$$

where $b: \mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}^n_x \to \mathbb{R}^n$ is a given vector fields (v.f.), and we will always suppose that

$$b \in C^0_c(\mathbb{R}^{n+1}, \mathbb{R}^n)\,.$$

We say that the (classical) well-posedness (WP) holds for (CP), if

$$\begin{array}{l} \forall\, x \in \mathbb{R}^n\, \exists\, \delta = \delta(x) > 0 \\ \qquad \text{and a unique } \gamma : (-\delta,\delta) \to \mathbb{R}^n \text{ solution of (CP)}\,. \end{array} \tag{WP}$$

Flow associated to a vector field

If (WP) holds for (CP), then it s well-known that $\exists$ a unique continuous flow $X : \mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}^n_x \to \mathbb{R}^n$ such that, for each $x \in \mathbb{R}^n$, $X(\cdot, x) : \mathbb{R} \to \mathbb{R}^n$ is a solution of (CP) and we will call X flow associated to b.

Rmk. Notice that, for each $t \in \mathbb{R}$,

$$X(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n \text{ is a homeomorphism.}$$

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Problem

Problem: Find out assumptions on b in order that:

(i) (WP) (or a weak form) holds for (CP), for weakly differentiable b in space (i.e., when $b(t,\cdot) \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$);

(ii) for each $t \in \mathbb{R}$, exists $q = q(t) \geq 1$ such that $X(t,\cdot) \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$.

Rmk. If $b \in L^\infty([0,T]; W^{1,\infty}(\mathbb{R}^n))$, from the classical Cauchy-Lipschitz theory, (WP) follows as well as that $X \in L^\infty([0,T]; W^{1,\infty}(\mathbb{R}^n))$.

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Some previous results

An interesting account of main results on this topic is

[AC, Ambrosio, Crippa, Proc.Roy.Soc. Edinburgh Sect. A, 2014]

Two fundamental results about (WP) **for weakly differentiable v.f.**

- [Di Perna, Lions,1989] Assume that

$$b \in C^0_c(\mathbb{R}^{n+1};\mathbb{R}^n) \cap L^1([0,T]; W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^n)) \,, \forall\, T > 0\,, \quad \text{(Sobreg)}$$

$$\operatorname{div}(b) \in L^1(\mathbb{R}; L^\infty(\mathbb{R}^n))\,. \quad \text{(Bddiv)}$$

  Then there exists a unique generalized flow $X : \mathbb{R}_t \times \mathbb{R}^n_x \to \mathbb{R}^n$ associated to b.

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Then there exists a unique generalized flow $X : \mathbb{R}_t \times \mathbb{R}^n_x \to \mathbb{R}^n$ associated to b.

Di Perna-Lions' notion of flow

$X : \mathbb{R}_t \times \mathbb{R}^n_x \to \mathbb{R}^n$ is said to be a generalized flow associated to b, according to Di Perna-Lions' theory, if $X \in C^0(\mathbb{R}_t; L^1_{\mathrm{loc}}(\mathbb{R}^n_x))$ and, for each $\beta \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ satisfying

$$\beta(X) \in L^\infty(\mathbb{R}; L^1_{\mathrm{loc}}(\mathbb{R}^n)), \ \sup_{z \in \mathbb{R}^n} |D\beta(z)| (1 + |z|) < \infty \,,$$

it holds

$$\begin{cases} \partial_t \beta(X) & = D\beta(X) \cdot b(t, X) \text{ on } (0,\infty) \times \mathbb{R}^n \\ \beta(X)(0,x) & = \beta(x) \quad \text{a.e. } x \in \mathbb{R}^n \end{cases} ,$$

where the equation must be understood in distributional sense.

Ambrosio's notion of flow

- [Ambrosio,2004] Assume that

$$b \in C_c^0(\mathbb{R}^{n+1};\mathbb{R}^n) \cap L^1([0,T]; BV_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^n))\,, \tag{BVreg}$$

$$\mathrm{div}\,(b) \in L^1([0,T]; L^\infty(\mathbb{R}^n))\,. \tag{Bddiv}$$

Then there exists a unique Lagrangian flow $X : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, according to Ambrosio's theory. More precisely, a map $X :\ [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be a regular Lagrangian flow associated to b if

(LF1) for a.e. $x \in \mathbb{R}^n$, the curve $\mathbb{R} \ni t \mapsto X(t,x)$ is an absolutely continuous solution of (CP);

(LF2) there exists a constant L, independent of t, such that

$$X(t,\cdot)_\#\mathcal{L}^n \le \ L\,\mathcal{L}^n\,.$$

Rmk. Di Perna-Lions and Ambrosio's notions of flow are equivalent provided that (Sobreg) and (Bddiv) hold.

Rmk. There are examples which prove that , if (BVreg) and (Bddiv) do not hold, then the well-posedness may fail.

Rmk. When $n = 1$, condition (Bddiv) turns out to the classical Lipschitz condition on b.

(WP) with unbounded div (b)

One of the first result of (WP) when (Bddiv) does not hold.

- [Desjardins,1996] Assume that (Sobreg) holds and

$$\exp(|\operatorname{div} b|) \in L^1(\mathbb{R}_t; L^1(\mathbb{R}^n_x)))$$

Then there exists a unique $X : \mathbb{R}_t \times \mathbb{R}^n_x \to \mathbb{R}^n$ such that $X \in L^\infty(\mathbb{R}_t \times \mathbb{R}^n_x, \mathbb{R}^n)$ such that, $\forall\, \Phi \in C^\infty_c(\mathbb{R}^n)$, it holds

$$\begin{cases} \partial_t \Phi(X) & = D\Phi(X) \cdot b(t, X) \text{ on } (0,\infty) \times \mathbb{R}^n \\ \Phi(X)(0,x) & = \Phi(x) \quad \text{a.e. } x \in \mathbb{R}^n \end{cases},$$

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where the equation must be understood in distributional sense.

(WP) and regularity of flows

Question: What about the Sobolev regularity of a flow?

Some counterexamples.

- [Jabin, 2016] For each $p \in [1, \infty)$ there exists a compactly supported, divergence free v.f. $b \in L^1([0,T]; W^{1,p}(\mathbb{R}^2))$, such that the associated regular Lagrangian flow X satisfies

$$X(t,\cdot) \notin W^{1,q}_{\mathrm{loc}}(\mathbb{R}^2) \text{ for each } q \geq 1, \text{ for each } t > 0\,.$$

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(WP) and regularity of flows

- [Alberti, Crippa, Mazzuccato, 2019] Let $n \geq 2$. Then there exists a compactly supported in space, v.f. $b \in L^\infty([0,\infty) \times \mathbb{R}^n_x) \cap L^1_{\rm loc}([0,\infty); W^{1,p}(\mathbb{R}^n))$, for each $p \in [1,\infty)$, such that the associated regular Lagrangian flow X satisfies

$$X(t,\cdot)^{-1} \notin W^{1,q}_{\rm loc}(\mathbb{R}^n) \text{ for each } q \geq 1, \text{ for each } t > 0\,.$$

(WP) and regularity with bounded div (b)

- [Clop, Jylhä, 2019] Let $ExpL(\Omega) := \{f : \Omega \to \mathbb{R} :\ \exp(|f|) \in L^1(\Omega)\}$. If $f \in ExpL(\Omega)$,

$$\|f\|_{Exp_\infty}(\Omega) := \inf\left\{\lambda > 0 : \int_\Omega \exp\left(\frac{|f(x)|}{\lambda}\right) dx < \infty\right\}.$$

Assume that b satisfy (Sobreg), (Bddiv) hold and $D_x b \in L^1([0,T]; ExpL_{\rm loc}(\mathbb{R}^n))$. Let $X : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a Lagrangian flow associated to b. Suppose that $q \geq 1$, $t > 0$ and a ball $B \subset \mathbb{R}^n$ satisfy

$$q < \frac{1}{2c_n I(t)} \text{ where } I(t) := \int_0^t \|D_x b(s,\cdot)\|_{Exp_\infty}(\tilde{B}) ds$$

and c_n is a suitable constant and $\tilde{B} = B_{3r}$ such that $X(s,B) \subset B_r$ for each $s \in [0,T]$. Then $X \in L^\infty([0,t]; W^{1,q}(B))$.

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- [BN,Brué, Nguyen, 2019] Assume that b satisfy (Sobreg), (Bddiv) and there exists $\beta > 0$ such that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^n} \exp\left(\beta|D_x b(t,x)|\right) dx < \infty . \qquad (\star)$$

  Let $X : [0, T] \times \mathbb{R}^n$ be the associated Lagrangian flow associated to b. Then, for each $t \in [0, T]$, $X(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism and $X(t,\cdot), X(t,\cdot)^{-1} \in W^{1,q_t}_{\mathrm{loc}}(\mathbb{R}^n)$, for any $0 \le t < \frac{\beta}{c_n}$, where $q_t := \frac{\beta}{C_1 t}$, for a suitable constant $c_n > 0$.

Rmk. Counterxamples are given in [BN] below integrability condition $(\star)$.

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(WP) and regularity with unbounded div (b)

- [Reimann,1976],[Clop, Jiang,Mateu, Orobitg,2018] Assume that (Sobreg) holds and

$$S_A b := \frac{D_x b + D_x b^T}{2} - \operatorname{div} b\,\mathbf{I}_{n\times n} \in L^1([0,T];L^\infty(\mathbb{R}^n)) \text{ and } n \geq 2\,.$$

Then there exists a unique Lagrangian flow $X : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ such that, for each $t\in[0,T]$, $X(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a *quasiregular* (or, equivalently, *quasiconformal*) homeomorphism, that is $X(t,\cdot)\in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^n)$ and there exists a constant $K = K(t) \geq 1$ such that $J_{X(t,\cdot)} := \det D_x X(t,\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$|D_x X(t,x)|^n \leq K\, J_{X(t,\cdot)}(x) \text{ for a.e. } x\in\mathbb{R}^n\,.$$

(WP) and regularity with unbounded div (b)

Rmk. If $S_A b(t,\cdot) \in L^\infty(\mathbb{R}^n)$ then $D_x b(t,\cdot) \in BMO(\mathbb{R}^n)$. In particular there exists $c > 0$ such that $\exp(c|D_x b(t,\cdot)|) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.
Rmk . It is well-known that there exists $q = q(t,n,K) > n$ such that $X(t,\cdot) \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^n)$.

- [Jiang, Li,Xiao, 2019] Assume that $n = 1$, (Sobreg) holds and

$$D_x b \in L^1([0,T]; BMO(\mathbb{R}))$$

Then there exist a unique(classical) flow $X : [0,T] \times \mathbb{R} \to \mathbb{R}$. Moreover for each $t \in [0,T]$, $X(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism, with $DxX(,t,) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$. Moreover $|D_x X(t,\cdot)| \in A_\infty(\mathbb{R})$. In particular $X(t,\cdot) \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$.

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Content

1. The problem
2. Some previous results
3. A new result

New result of (WP) and regularity with unbounded div (b)

In the following we will denote by B_R a closed ball of $\mathbb{R}^n$ centered at a given point x_0 and radius $R > 0$.

Theorem([Ambrosio, Nicolussi Golo, S.C.,2020)]

Let $n \geq 1$ and $b \in C^0(\mathbb{R}^{n+1}, \mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}_t, W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n_x, \mathbb{R}^n))$ with

$$\mathrm{spt}(b) \subset Q_{T_0,R_0} := [-T_0, T_0] \times B_{R_0} .$$

Assume there exists $p > n$ such that

$$\exp\left(2\,T_0\,\frac{p^2}{p-n}\,|D_x b|\right) \in L^1(Q_{T_0,R_0}) . \tag{Expint}$$

Then there exists a unique continuous flow $X : \mathbb{R}_t \times \mathbb{R}^n_x \to \mathbb{R}^n$ associated to b satisfying the following properties:

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New result of (WP) and regularity with unbounded div (b)

(i) for each $t \in \mathbb{R}$, $X(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism and

$$X(t,\cdot),\, X(t,\cdot)^{-1} \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^n)\,, \tag{1}$$

and, for each $R \geq R_0 + 1$, there exists a positive constant $C_1 = C_1(n, T_0, p)$ such that

$$\begin{aligned}\int_{B_R} |D_x X(t,x)|^p\, dx \leq\ & C_1 \left\| \exp\left(2\, T_0\, \frac{p^2}{p-n}\, |D_x b|\right) \right\|_{L^1(Q_{T_0,R})} \\ =\ & C_1 \left\| \exp\left(2\, T_0\, \frac{p}{p-n}\, |D_x b|\right) \right\|_{L^p(Q_{T_0,R})} \\ & \forall\, t \in [-T_0, T_0]\,.\end{aligned} \tag{SE}$$

(ii) $J_X(t,\cdot) := \det D_x X(t,\cdot)$ satisfies the following properties:

$$\text{for each } t \in \mathbb{R},\ J_X(t,\cdot) \in L^{\frac{p}{n}}_{\rm loc}(\mathbb{R}^n)\,; \tag{2}$$

for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n_x$, for each $T > 0$
$[-T,T] \ni t \mapsto \mathrm{div}\,(b)(t,X(t,x))$ is $L^r(-T,T)\ \forall\, r \in [1,\infty)$

$$\text{and } J_X(t,x) = \exp\left(\int_0^t \mathrm{div}\,(b)(v,X(v,x))\,dv\right) \quad \forall\, t \in [-T,T]\,; \tag{3}$$

$$\text{for each } t \in \mathbb{R},\ X(t,\cdot)_{\#}\mathcal{L}^n = J_{X^{-1}(t,\cdot)}\,\mathcal{L}^n\,. \tag{4}$$

Proof's outline of (SE)

1st step. Estimate (Expint)on b implies that Osgood's uniqueness criterion for ODEs applies. As a consequence, we can infer that the classical (WP) holds for b. Then there exists a unique continuous flow $X : \mathbb{R}_t \times \mathbb{R}^n_x \to \mathbb{R}^n$.

2nd step. Let $(\rho_\epsilon)_\epsilon$ be a family of mollifiers and let $b_\epsilon(t,x) := (b(t,\cdot) * \rho_\epsilon)(x)$. Let X_ϵ be the (regular) flow associated to b_ϵ. Then one can prove that, for each $t \in \mathbb{R}$,

$$X_\epsilon(t,\cdot) \to X(t,\cdot) \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}^n_x), \text{ as } \epsilon \to 0^+ .$$

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3rd (key) step. By classical properties of ODE's flow and Hölder inequality, one can prove that Sobolev estimate (SE) holds for X_ϵ.

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Applications to PDEs

From this result of Sobolow regularity for flows, we have been studying possible applications to the regularity of weak solutions of the Cauchy problem for the transport and continuity equations. Namely

$$\begin{cases} \partial_t u + \ b \cdot \nabla u & = \ 0 \text{ in } (0,T) \times \mathbb{R}^n_x \\ u(0,\cdot) & = \ \bar{u} \end{cases}, \tag{CPTE}$$

$$\begin{cases} \partial_t \rho + \operatorname{div}(b\rho) & = \ 0 \text{ in } (0,T) \times \mathbb{R}^n \\ \rho(0,\cdot) & = \ \bar{\rho} \end{cases} \tag{CPCE}$$

meant in sense of distributions.