Long time existence of solutions to the Kirchhoff equation

Tokio Matsuyama
Department of Mathematics
Chuo Univertsity
Tokyo, Japan
Joint work with Prof. Michael Ruzhansky (Ghent University)

Dispersive and subelliptic PDEs
10 February 2020
Centro di Ricerca Matematica, Ennio De Giorgi, Scuola Normale Superiore

§1 Kirchhoff equation

In 1883 G. Kirchhoff proposed the equation

$$\partial_t^2 u - \left(1 + \int_\Omega |\nabla u(t,y)|^2 \, dy\right) \Delta u = 0 \quad (t \in \mathbb{R},\ x \in \Omega)$$

as a model equation for transversal motion of the elastic string, where Ω is a domain in $\mathbb{R}^n$.

G. Kirchhoff, *Vorlesungen über Mechanik*, Teubner, Leibzig (1883)

Known results on global existence theorems

1. Analytic class:
   - Bernstein ('40): 1-dim
   - Pohožhaev ('75) extends the result of Bernstein to several space dimensions.

Known results on global existence theorems

1. Analytic class:
   - Bernstein ('40): 1-dim
   - Pohožhaev ('75) extends the result of Bernstein to several space dimensions.
2. Degenete case:
   - Arosio and Spagnolo ('84)
   - D'Ancona and Spagnolo ('94)
   - Kajitani and Yamaguti ('94)

Known results on global existence theorems

1. Analytic class:
   - Bernstein ('40): 1-dim
   - Pohožhaev ('75) extends the result of Bernstein to several space dimensions.
2. Degenete case:
   - Arosio and Spagnolo ('84)
   - D'Ancona and Spagnolo ('94)
   - Kajitani and Yamaguti ('94)
3. Quasi-analytic classes: Intermediate spaces between analytic class and Sobolev spaces ($\neq \gamma^s_{L^2}$)
   - Nishihara ('84)
   - Ghisi and Gobbino (2011)

Known results on global existence theorems

1. Analytic class:
   - Bernstein ('40): 1-dim
   - Pohožhaev ('75) extends the result of Bernstein to several space dimensions.
2. Degenete case:
   - Arosio and Spagnolo ('84)
   - D'Ancona and Spagnolo ('94)
   - Kajitani and Yamaguti ('94)
3. Quasi-analytic classes: Intermediate spaces between analytic class and Sobolev spaces ($\neq \gamma^s_{L^2}$)
   - Nishihara ('84)
   - Ghisi and Gobbino (2011)
4. Another classes:
   - Manfrin (2005): spectral gap

Known results on global existence theorems

1. Analytic class:
   - Bernstein ('40): 1-dim
   - Pohožhaev ('75) extends the result of Bernstein to several space dimensions.
2. Degenete case:
   - Arosio and Spagnolo ('84)
   - D'Ancona and Spagnolo ('94)
   - Kajitani and Yamaguti ('94)
3. Quasi-analytic classes: Intermediate spaces between analytic class and Sobolev spaces ($\neq \gamma^s_{L^2}$)
   - Nishihara ('84)
   - Ghisi and Gobbino (2011)
4. Another classes:
   - Manfrin (2005): spectral gap
5. Sobolev well-posedness for small data:
   Greenberg and Hu ('80), D'Ancona and Spagnolo ('93, '94), Kajitani (2012), M. (2010), Ruzhansky and M. (2013), Racke ('95), and Yamazaki ('95, 2004, 2005)

§2 Open problems

1. Global H^σ-well-posedness for $\sigma \geq 3/2$ without any smallness condition on data
2. Blow-up theorem in $H^{\frac{3}{2}} \times H^{\frac{1}{2}}$
3. Existence of local H^σ-solution for $1 \leq \sigma < 3/2$
4. Gevrey well-posedness

§2 Open problems

1. Global H^σ-well-posedness for $\sigma \geq 3/2$ without any smallness condition on data
2. Blow-up theorem in $H^{\frac{3}{2}} \times H^{\frac{1}{2}}$
3. Existence of local H^σ-solution for $1 \leq \sigma < 3/2$
4. Gevrey well-posedness

Today's talk is about long time existence of Gevrey space solutions.

§3 Result

Consider the Cauchy problem

$$\begin{cases} \partial_t^2 u - \left(1 + \int_{\mathbb{R}^n} |\nabla u(t,y)|^2 \, dy\right) \Delta u = 0, & t > 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.1}$$

Kirchhoff equation has a first integral:

$$\mathscr{H}(u;t) = \mathscr{H}(u;0) \quad \text{as long as solution exists,} \tag{2.2}$$

where

$$\mathscr{H}(u;t) = \frac{1}{2}\left\{\|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2\right\} + \frac{1}{4}\|\nabla u(t)\|_{L^2}^4.$$

Gevrey class of L^2 type

- Gevrey-Roumieu class: $\gamma^s_{L^2} = \gamma^s_{L^2}(\mathbb{R}^n)$;

$$\gamma^s_{L^2} = \bigcup_{\eta>0} \gamma^s_{\eta,L^2},$$

where $f \in \gamma^s_{\eta,L^2}$ if

$$\int_{\mathbb{R}^n} e^{\eta|\xi|^{\frac{1}{s}}} |\hat{f}(\xi)|^2 \, d\xi < \infty.$$

In particular case $s = 1$, $\gamma^1_{L^2}$ is analytic class $\mathcal{A}_{L^2}$, where

$$f \in \mathcal{A}_{L^2}$$
$$\Longleftrightarrow \exists\, A, C \geq 0 \,\text{such that}\, \|\partial^\alpha f\|_{L^2} \leq CA^{|\alpha|}\alpha!, \quad \forall\, \alpha \in (\mathbb{N} \cup \{0\})^n.$$

Theorem 2.1 (Ruzhansky and M.: J. Anal. Math. (2019))

Let $T > 0$ and $s > 1$. Let $M > 2$, $R > 0$ and denote

$$\eta_0(M, R, T) = 2sM^2 e^{4M^2} RT^{1+\frac{1}{s}} + 4M^2.$$

If the functions $u_0, u_1 \in \gamma^s_{\eta, L^2}$, for some $\eta > \eta_0(M, R, T)$, satisfy conditions

$$2\mathscr{H}(u; 0) < \frac{M^2}{4} - 1,$$

$$\left\| \left((-\Delta)^{\frac{3}{4}} u_0, (-\Delta)^{\frac{1}{4}} u_1 \right) \right\|^2_{\gamma^s_{\eta,L^2} \times \gamma^s_{\eta,L^2}} \leq R,$$

then the Cauchy problem (2.1) *admits a unique solution*
$u \in C^1([0, T]; \gamma^s_{L^2})$.

We note that Theorem 2.1 does not seem to require the smallness of data. In fact, M and R (measuring the size of the data) are allowed to be large. However, it follows that η (measuring the regularity of the data) then also have to be big. So, we can informally describe conditions of Theorem 2.1 that 'the larger the data is the more regular it has to be'.

§4 Energy estimates for linear Cauchy problem

Let us consider the linear Cauchy problem

$$\begin{cases} \partial_t^2 v - c(t)^2 \Delta v = 0, & t \in (0, T), \quad x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), \quad \partial_t v(0,x) = v_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.1}$$

Assume that

$$c(t) \in \mathrm{Lip}_{\mathrm{loc}}([0, T)),$$

$$1 \le c(t) \le M, \quad t \in [0, T], \tag{3.2}$$

$$\left| c'(t) \right| \le \frac{K}{(T-t)^q}, \quad a.e.\, t \in [0, T) \tag{3.3}$$

for some $q > 1$, $M > 1$ and $K > 0$.

Proposition 3.1 (Cf. Colombini, Del Santo and Kinoshita, Ann. Sc. Norm. Sup. Pisa (2002))

Let $1 \le s < \frac{q}{q-1}$ *for some* $q > 1$. *Assume that* $c(t) \in \mathrm{Lip}_{\mathrm{loc}}([0, T))$ *satisfies* (3.2) *and* (3.3). *If* $(v_0, v_1) \in (-\Delta)^{-\frac{3}{4}}\gamma^s_{\eta,L^2} \times (-\Delta)^{-\frac{1}{4}}\gamma^s_{\eta,L^2}$ *for some* η *satisfying*

$$\eta > \frac{2K}{q-1} + 4M^2,$$

then the linear Cauchy problem (3.1) *admits a unique solution* $v \in \bigcap_{j=0}^{1} C^j([0, T]; (-\Delta)^{-\frac{3}{4}+\frac{j}{2}}\gamma^s_{\eta',L^2})$ *s.t.*

$$\begin{aligned} &\|(-\Delta)^{\frac{3}{4}} v(t)\|^2_{\gamma^s_{\eta',L^2}} + \|(-\Delta)^{\frac{1}{4}} \partial_t v(t)\|^2_{\gamma^s_{\eta',L^2}} \\ \le& M^2 e^{4M^2 \max\{1, T^{1-(qs-s)}\}} \|((-\Delta)^{\frac{3}{4}} v_0, (-\Delta)^{\frac{1}{4}} v_1)\|^2_{\gamma^s_{\eta,L^2} \times \gamma^s_{\eta,L^2}} \end{aligned} \tag{3.4}$$

for $t \in [0, T]$, *where* $\eta' = \eta - \left(\frac{2K}{q-1} + 4M^2\right) > 0$.

§5 Proof of Theorem 3.1

Consider the *linear* Cauchy problem in the strip $(0, T) \times \mathbb{R}^n$:

$$\partial_t^2 v - c(t)^2 \Delta v = 0, \quad t \in (0, T), \quad x \in \mathbb{R}^n, \tag{3.5}$$

with initial condition

$$v(0, x) = u_0(x), \quad \partial_t v(0, x) = u_1(x). \tag{3.6}$$

Here $c(t)$ belongs to a class $\mathscr{K}(T)$ defined as follows:

Class $\mathscr{K}(T)$. *Given constants*

$$q > 1, \quad M > 2 \quad \textit{and} \quad K_0 > 0,$$

we say that $c(t) \in \mathscr{K}(T) = \mathscr{K}(q, M, K_0, T)$ *if*

$$c(t) \in \mathrm{Lip}_{\mathrm{loc}}([0, T)),$$

$$1 \le c(t) \le M, \quad t \in [0, T],$$

$$\left|c'(t)\right| \le \frac{K_0}{(T-t)^q}, \quad a.e.\ t \in [0, T).$$

We use the following:

Theorem (Schauder-Tychonoff). Let X be a Fréchet space, and $\mathcal{K}$ a nonempty, convex and compact subset of X. If

$$\Theta : \mathcal{K} \longrightarrow \mathcal{K}$$

is continuous, then Θ has a fixed point in $\mathcal{K}$.

By the energy estimate (3.4) from Proposition 3.1, if

$$(u_0, u_1) \in (-\Delta)^{-\frac{3}{4}}\gamma^s_{\eta,L^2} \times (-\Delta)^{-\frac{1}{4}}\gamma^s_{\eta,L^2}$$

for all η satisfying

$$\eta > \frac{2K_0}{q-1} + 4M^2,$$

then the Cauchy problem (3.5)–(3.6) admits a unique solution v satisfying

$$v \in \bigcap_{j=0}^{1} C^j\left([0,T]; (-\Delta)^{-\frac{3}{4}+\frac{j}{2}}\gamma^s_{\eta',L^2}\right), \tag{3.7}$$

provided that $1 \le s < \frac{q}{q-1}$ and $q > 1$, where η' is the real number fulfilled with

$$\eta' = \eta - \left(\frac{2K_0}{q-1} + 4M^2\right) > 0. \tag{3.8}$$

We define the functional

$$\tilde{c}(t) = \sqrt{1 + \int_{\mathbb{R}^n} |\nabla v(t,x)|^2 \, dx}.$$

This defines the mapping

$$\Theta : c(t) \mapsto \tilde{c}(t).$$

We can show that

$$\mathcal{K}(T)\text{: convex and compact in } L^\infty_{\mathrm{loc}}([0,T)).$$

If we show that

$$\Theta \text{ maps continuously from } \mathcal{K}(T) \text{ into itself,}$$

Schauder-Tychonoff fixed point theorem allows us to conclude the proof.

Proposition 3.2

Let $M > 2$, $T > 0$ *and* $R > 0$. *Let* $1 < q < 2$ *and* $s > 1$ *be such that*

$$s = \frac{1}{q-1}. \tag{3.9}$$

Let $\eta > 0$ *be such that*

$$\eta > \frac{2M^2 e^{4M^2} R T^q}{q-1} + 4M^2. \tag{3.10}$$

If $(u_0, u_1) \in (-\Delta)^{-\frac{3}{4}} \gamma^s_{\eta, L^2} \times (-\Delta)^{-\frac{1}{4}} \gamma^s_{\eta, L^2}$ *satisfy*

$$2\mathscr{H}(v; 0) \le \frac{M^2}{4} - 1, \tag{3.11}$$

$$\left\| ((-\Delta)^{\frac{3}{4}} u_0, (-\Delta)^{\frac{1}{4}} u_1) \right\|^2_{\gamma^s_{\eta, L^2} \times \gamma^s_{\eta, L^2}} \le R, \tag{3.12}$$

then, setting

$$K_0 = M^2 e^{4M^2} R T^q, \tag{3.13}$$

we have the following statement: For any $c(t) \in \mathscr{K}(T)$, let v be a solution to the Cauchy problem (3.5)–(3.6) satisfying (3.7). Then

$$1 \le \tilde{c}(t) \le M, \quad t \in [0, T], \tag{3.14}$$

$$\left|\tilde{c}'(t)\right| \le \frac{K_0}{(T-t)^q}, \quad t \in [0, T). \tag{3.15}$$

we have the following statement: For any $c(t) \in \mathscr{K}(T)$, let v be a solution to the Cauchy problem (3.5)–(3.6) satisfying (3.7). Then

$$1 \le \tilde{c}(t) \le M, \quad t \in [0, T], \tag{3.14}$$

$$\left|\tilde{c}'(t)\right| \le \frac{K_0}{(T-t)^q}, \quad t \in [0, T). \tag{3.15}$$

$$\text{Proposition 3.2} \Longrightarrow \Theta : \mathcal{K}(T) \longrightarrow \mathcal{K}(T)$$

Thank you for your attention !!