# Very weak solutions to wave equations on graded Lie groups

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# Plan

- Very weak solutions to wave equations on graded groups:

M. Ruzhansky and N. Yessirkegenov.
*Very weak solutions to hypoelliptic wave equations.*
J. Differential Equations, **268** (2020), 2063–2088.

- Global well-posedness for a semilinear heat equation on unimodular Lie groups:

M. Ruzhansky and N. Yessirkegenov.
*Existence and non-existence of global solutions for semilinear heat equations and inequalities on sub-Riemannian manifolds, and Fujita exponent on unimodular Lie groups.*
arXiv:1812.01933, (2018).

# Motivation

We want to investigate the well-posedness of, for example,

$$\boxed{\begin{cases} \partial_t^2 u(t,x) - a(t)\Delta u(t,x) = 0 \\ u(0,x) = u_0(x), \partial_t u(0,x) = u_1(x) \end{cases}}$$

where $a(t)$ is the Heaviside or Delta functions.

Several questions:

• how to interpret the equation when $u$ is a distribution?
(recall e.g. Schwartz' impossibility result and Hörmander's wave front conditions)

• what is the right notion of well-posedness for equations with such irregular coefficients?
(if $u$ is smooth, the equation makes sense; but if $u$ has singularities, notions of weak solutions or distributional solutions do not work ...)

# Different regularities of coefficients

For example, consider the wave equation

$$\partial_t^2 u - a(t)\partial_x^2 u = 0$$

- $a \in C^1$: classical or distributional solutions depending on Cauchy data;
- $a \in C^\alpha$, $0 < \alpha < 1$: Gevrey/ultradistributional well-posedness;
- $a \in L^1_{loc}$: solution in terms of Fourier hyperfunctions;
- $a$ is measurable: Kato theory of semigroups;
- $a \in \mathcal{D}'$: ?? our question;
- $a$ is Colombeau: Colombeau solutions for some examples (e.g. Lafon+Oberguggenberger, Hörmann, Marti, ...);

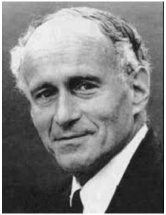
If $a$ is a distribution, known 'regular' approaches are not enough, and Colombeau approach may be too abstract.

Ideally, from a good notion of a solution, we want that:

- it is weak enough to guarantee that solutions exist;
- it is strong enough to give uniqueness and consistency with classical/distributional solutions if they exist.

# Weakening of the notion of 'solution' to assure existence

Historically, the development of PDEs consisted in weakening the notion of



'solution' to ensure its existence:

- classical solutions: pointwise solutions when everything is regular;
- Sobolev's weak solutions when everything is in $L^1_{loc}$;
- Schwartz' distributional solutions for interpretation as functionals;
- ultradistributional solutions when no distributional well-posedness (e.g. already for wave equations with nonnegative smooth propagation speed and smooth data: *Colombini & Spagnolo, Acta Math, 1982*);

All these notions are backwards compatible (consistency).

Our 'very weak' solutions for settings where the theory of distributions is not working. Crucial part: **backward compatibility (consistency).**

# Solutions have been always understood in different ways

Different types of notions of solutions:

- classical solutions: pointwise, same notion for all equations;
- Sobolev's weak solutions:
  ☛☛ the notion of solutions depends on equation!
- distributional solutions: Schwartz' theory exists independently of equations; same for hyperfunctions;
- Colombeau solutions: Colombeau's theory exists independently of equations; Also, problems with backward compatibility?!

✌ Very weak solutions: for distributional coefficients and data, and also allowing multiplicities (Ruzhansky+Garetto, Arch. Ration. Mech. Anal., 2015). ☛ similarly to weak solution,
☛☛ it is adapted to the equation
(properties of solutions to the regularised equation)

# Simple example

Wave equation with $a = a(t) \geq 0$ (for simplicity assume even $a > 0$):

$$\text{(CP)} \qquad \begin{cases} \partial_t^2 u - a(t)\partial_x^2 u = 0, \ t \in [0,T], \ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \partial_t u(0,x) = u_1(x). \end{cases}$$

Fact: $u_0, u_1 \in C_0^\infty$, $a \in C$, but there may be no distributional solution $u(t,\cdot) \notin \mathcal{D}'(\mathbb{R}), \forall t > 0$. However, if $a \in C^\infty$, there is solution $u \in C^\infty$.

Regularisation: consider Friedrichs-type regularisations $a_\varepsilon = a * \psi_{\omega(\varepsilon)}$ where $\psi_{\omega(\varepsilon)}(t) = \omega(\varepsilon)^{-1}\psi(t/\omega(\varepsilon))$, and $\omega(\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$.

A general family $\{f_\varepsilon\}_{\varepsilon>0} \in C^\infty$ is called moderate if $\forall \alpha \in \mathbb{N}$ $\exists N \in \mathbb{N}, c > 0$:

$$\sup_{x\in\mathbb{R}} |\partial^\alpha f_\varepsilon(x)| \leq c\varepsilon^{-N}, \quad \forall \varepsilon \in (0,1].$$

NOTE: if $a \in \mathcal{E}'(\mathbb{R})$, then the family $\{a_\varepsilon = a * \psi_{\omega(\varepsilon)}\}_{\varepsilon>0}$ is moderate. So we can think of distributions as moderate-families:

compactly supported distributions $\mathcal{E}'(\mathbb{R}) \subset \{C^\infty\text{-moderate families}\}$

# Simple example of a very weak solution

**Definition (Simplest case)**

The net $(u_\varepsilon)_\varepsilon \in C^\infty([0,T]\times\mathbb{R})$ is a very weak solution of (CP) if there exists a moderate regularisation $a_\varepsilon$ of the coefficient $a$ such that $(u_\varepsilon)_\varepsilon$ solves the regularised Cauchy problem

$$\begin{cases} \partial_t^2 u_\varepsilon(t,x) - a_\varepsilon(t)\partial_x^2 u_\varepsilon(t,x) = 0, \\ u_\varepsilon(0,x) = u_0, \partial_t u_\varepsilon(0,x) = u_1, \end{cases}$$

for all $\varepsilon \in (0,1]$, and $(u_\varepsilon)_\varepsilon$ is moderate

**Summary of results for this notion:**

- (existence) for any $a \in \mathcal{E}'$, the Cauchy problem (CP) has a very weak solution;
- (uniqueness) a very weak solution is unique in an appropriate sense;
- (consistency) with classical or distributional/ultradistributional solutions:

$*$ $\boxed{\text{if } a \in C^\infty \text{ and } u_0, u_1 \in C^\infty}$ so that there exists a classical solution $u \in C^\infty$, then for any regularising family $a_\varepsilon$ for $a$, we have $u_\varepsilon \to u$ in $C^\infty$ as $\varepsilon \to 0$

$*$ $\boxed{\text{if } a \in C^\infty \text{ and } u_0, u_1 \in \mathcal{D}'}$ so that there exists a distributional solution $u(t) \in \mathcal{D}'$, then $u_\varepsilon \to u$ in $\mathcal{D}'$ as $\varepsilon \to 0$

# Wave equation for hypoelliptic operators

In this talk we are interested in the well-posedness of the following Cauchy problem for general hypoelliptic (Rockland operator of homogeneous degree $\nu$) differential operator $\mathcal{R}$ on general graded Lie group $\mathbb{G}$ with the non-negative propagation speed $a = a(t)$:

$$\begin{cases} \partial_t^2 u(t) + a(t)\mathcal{R}u(t) = 0, \ t \in [0, T], \\ u(0) = u_0 \in L^2(\mathbb{G}), \\ \partial_t u(0) = u_1 \in L^2(\mathbb{G}). \end{cases} \tag{1}$$

A (connected and simply connected) Lie group $\mathbb{G}$ is **graded** if its Lie algebra:

$$\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i,$$

where $\mathfrak{g}_1, \mathfrak{g}_2, \ldots$, are vector subspaces of $\mathfrak{g}$, only finitely many not $\{0\}$, and

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \ \ \forall i, j \in \mathbb{N}.$$

If $\mathfrak{g}_1$ generates $\mathfrak{g}$ through commutators, the group is said to be **stratified**, and the sum of squares of a basis of vector fields in $\mathfrak{g}_1$ yields a sub-Laplacian on $\mathbb{G}$.

**Example 1 (Abelian case)**. The abelian group $(\mathbb{R}^n, +)$ is graded: its Lie algebra $\mathbb{R}^n$ is trivially graded, i.e. $\mathfrak{g}_1 = \mathbb{R}^n$.

**Example 2 (Heisenberg group)**. The Heisenberg group $\mathbb{H}_n$ is stratified: its Lie algebra $\mathfrak{h}_n$ can be decomposed as $\mathfrak{h}_n = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_1 = \oplus_{j=1}^{n} \mathbb{R}X_j \oplus \mathbb{R}Y_j$ and $\mathfrak{g}_2 = \mathbb{R}T$, where

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_t, \ \ Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_t, \ \ j = 1, \ldots, n, \ \ T = \partial_t. \qquad (2)$$

# Positive Rockland operators

Recall: Rockland operator $\mathcal{R}$ is a homogeneous hypoelliptic invariant differential operator on a nilpotent Lie group. (after Helffer and Nourrigat)
To give some examples, this setting includes:

- for $\mathbb{G} = \mathbb{R}^n$, $\mathcal{R}$ may be any positive homogeneous elliptic differential operator with constant coefficients. For example, we can take

$$\mathcal{R} = (-\triangle)^m \text{ or } \mathcal{R} = (-1)^m \sum_{j=1}^{n} a_j \left(\frac{\partial}{\partial x_j}\right)^{2m}, \ a_j > 0, \ m \in \mathbb{N};$$

- for $\mathbb{G} = \mathbb{H}_n$ the Heisenberg group, we can take

$$\mathcal{R} = (-\mathcal{L})^m \text{ or } \mathcal{R} = (-1)^m \sum_{j=1}^{n} (a_j X_j^{2m} + b_j Y_j^{2m}), a_j, b_j > 0, m \in \mathbb{N},$$

  where $\mathcal{L}$ is the sub-Laplacian and $X_j, Y_j$ are the left invariant vector fields.

- for any stratified Lie group (or homogeneous Carnot group) with vectors $X_1, \dots, X_k$ spanning the first stratum, we can take

$$\mathcal{R} = (-1)^m \sum_{j=1}^{k} a_j X_j^{2m}, \ a_j > 0,$$

so that in particular, for $m = 1$, $\mathcal{R}$ is a positive sub-Laplacian;

- for any graded Lie group $\mathbb{G} \sim \mathbb{R}^n$ with dilation weights $\nu_1, \dots, \nu_n$ let us fix the basis $X_1, \dots, X_n$ of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ satisfying

$$D_r X_j = r^{\nu_j} X_j, \ j = 1, \dots, n, r > 0,$$

where $D_r$ denote dilations on the Lie algebra. If $\nu_0$ is any common multiple of $\nu_1, \dots, \nu_n$, the operator

$$\mathcal{R} = \sum_{j=1}^{n} (-1)^{\frac{\nu_0}{\nu_j}} a_j X_j^{2\frac{\nu_0}{\nu_j}}, a_j > 0$$

is a Rockland operator of homogeneous degree $2\nu_0$.

# Some discussion of Rockland operators

- Rockland operator $\mathcal{R}$ is a homogeneous hypoelliptic invariant differential operator on a nilpotent Lie group.
- $\mathcal{R}$ exists $\Leftrightarrow$ dilation weights are rational $\Leftrightarrow$ the group is graded.
- Folland and Stein: this is a natural setting to combine harmonic analysis & PDEs. (and the usual working assumption)
- Rockland condition: for every representation $\pi \in \widehat{\mathbb{G}}$, except for the trivial representation, the operator $\pi(\mathcal{R})$ is injective on $\mathcal{H}_\pi^\infty$, that is,

$$\forall v \in \mathcal{H}_\pi^\infty, \;\; \pi(\mathcal{R})v = 0 \Rightarrow v = 0.$$

  Here $\pi(\mathcal{R}) := d\pi(\mathcal{R})$ is the infinitesimal representation of the Rockland operator $\mathcal{R}$ as of an element of the universal enveloping algebra of $\mathbb{G}$.
- Rockland, R. Beals, Helffer-Nourrigat (1979): equivalence of above conditions

# Wave equation for hypoelliptic operators

Let $\mathcal{R}$ be a Rockland operator on a graded Lie group $G$.

$$\partial_t^2 u(t) + a(t)\mathcal{R}u(t) = f(t),\ u(0) = u_0 \in L^2(G),\ \partial_t u(0) = u_1 \in L^2(G).$$

- Ruzhansky+Tokmagambetov, ARMA 2017: for $\mathcal{R}$ with discrete spectrum; However, the Rockland operator $\mathcal{R}$ has continuous spectrum.

We will use $\mathcal{R}$-Gevrey (Roumieu) $\mathcal{G}^s_{\mathcal{R}}(G)$ and $\mathcal{R}$-Gevrey (Beurling) type spaces $\mathcal{G}^{(s)}_{\mathcal{R}}(G)$ for $s \geq 1$, which are defined by

$$\mathcal{G}^s_{\mathcal{R}}(G) := \{f \in C^\infty(G) | \exists A > 0 : \|e^{A\mathcal{R}^{\frac{1}{2s}}} f\|_{L^2(G)} < \infty\}$$

and

$$\mathcal{G}^{(s)}_{\mathcal{R}}(G) := \{f \in C^\infty(G) | \forall A > 0 : \|e^{A\mathcal{R}^{\frac{1}{2s}}} f\|_{L^2(G)} < \infty\}.$$

Subelliptic Sobolev spaces $\|f\|_{H^s_{\mathcal{R}}(G)} := \|(I+\mathcal{R})^{\frac{s}{\nu}} f\|_{L^2(G)}$.

For compact Lie groups, such spaces were studied in Ruzhansky+Garetto (JDE, 2017)

**Theorem (Ruzhansky+Taranto)**

$\mathcal{R}$ - a positive Rockland operator of homogeneous degree $\nu$, $T>0$. Then

- Let $a\in \mathrm{Lip}([0,T])$ with $a(t)\geq a_0>0$. Let $s\in\mathbb{R}$, $(u_0,u_1)\in H^{s+\frac{\nu}{2}}_{\mathcal{R}}(G)\times H^s_{\mathcal{R}}(G)$. Then $\exists!$ solution satisfying
$$\|u(t,\cdot)\|^2_{H^{s+\frac{\nu}{2}}_{\mathcal{R}}(G)}+\|\partial_t u(t,\cdot)\|^2_{H^s_{\mathcal{R}}(G)}\leq C(\|u_0\|^2_{H^{s+\frac{\nu}{2}}_{\mathcal{R}}(G)}+\|u_1\|^2_{H^s_{\mathcal{R}}(G)});$$
- Let $a\in C^\alpha([0,T])$ with $0<\alpha<1$ and $a(t)\geq a_0>0$. If $(u_0,u_1)\in \mathcal{G}^s_{\mathcal{R}}(G)\times\mathcal{G}^s_{\mathcal{R}}(G)$, then $\exists!$ solution in $C^2([0,T],\mathcal{G}^s_{\mathcal{R}}(G))$ provided that
$$1\leq s<1+\frac{\alpha}{1-\alpha};$$
- Let $a\in C^\ell([0,T])$ with $\ell\geq 2$ and $a(t)\geq 0$. If $(u_0,u_1)\in \mathcal{G}^s_{\mathcal{R}}(G)\times\mathcal{G}^s_{\mathcal{R}}(G)$, then $\exists!$ solution in $C^2([0,T],\mathcal{G}^s_{\mathcal{R}}(G))$, provided that
$$1\leq s<1+\frac{\ell}{2};$$
- Let $a\in C^\alpha([0,T])$ with $0<\alpha<2$ and $a(t)\geq 0$. If $(u_0,u_1)\in \mathcal{G}^s_{\mathcal{R}}(G)\times\mathcal{G}^s_{\mathcal{R}}(G)$, then $\exists!$ solution in $C^2([0,T],\mathcal{G}^s_{\mathcal{R}}(G))$, provided that
$$1\leq s<1+\frac{\alpha}{2}.$$

Now we shall describe the notion of very weak solutions and formulate the corresponding results for distribution $a \in \mathcal{D}'([0,T])$. First, we regularise the distributional coefficient $a$ by the convolution with a suitable mollifier $\psi$ obtaining families of smooth function $(a_\varepsilon)_\varepsilon$ as follows

$$a_\varepsilon = a * \psi_{\omega(\varepsilon)} \tag{3}$$

with

$$\psi_{\omega(\varepsilon)}(t) = (\omega(\varepsilon))^{-1}\psi(t/\omega(\varepsilon)),$$

where $\omega(\varepsilon) > 0$ (which we will choose later) is such that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$, and $\psi$ is a Friedrichs-mollifier, that is,

$$\psi \in C_0^\infty(\mathbb{R}), \quad \psi \geq 0 \text{ and } \int \psi = 1.$$

It turns out that the net $(a_\varepsilon)_\varepsilon$ is $C^\infty$-moderate, in the sense that their $C^\infty$-seminorms can be estimated by a negative power of $\varepsilon$. More precisely, we will make use of the following notions of moderateness.

## Definition

- A net of functions $(f_\varepsilon)_\varepsilon \in C^\infty(\mathbb{R})^{(0,1]}$ is said to be $C^\infty$- moderate if for all $K \Subset \mathbb{R}$ and for all $\alpha \in \mathbb{N}_0$ there exist $N \in \mathbb{N}_0$ and $c > 0$ such that $\sup_{t\in K} |\partial^\alpha f_\varepsilon(t)| \leq c\varepsilon^{-N-\alpha}$, for all $\varepsilon \in (0,1]$, where $K \Subset \mathbb{R}$ means that $K$ is a compact set in $\mathbb{R}$.
- A net of functions $(u_\varepsilon)_\varepsilon \in C^\infty([0,T]; H^s_{\mathcal{R}})^{(0,1]}$ is $C^\infty([0,T]; H^s_{\mathcal{R}})$-moderate if there exist $N \in \mathbb{N}_0$ and $c_k > 0$ for all $k \in \mathbb{N}_0$ such that $\|\partial_t^k u_\varepsilon(t,\cdot)\|_{H^s_{\mathcal{R}}} \leq c_k \varepsilon^{-N-k}$, for all $t \in [0,T]$ and $\varepsilon \in (0,1]$.
- We say that a net of functions $(u_\varepsilon)_\varepsilon \in C^\infty([0,T]; H^{-\infty}_{(s)})^{(0,1]}$ is $C^\infty([0,T]; H^{-\infty}_{(s)})$-moderate if there exists $\eta > 0$ and, for all $p \in \mathbb{N}_0$ there exists $c_p > 0$ and $N_p > 0$ such that $\|e^{-\eta\mathcal{R}^{\frac{1}{2s}}}\partial_t^p u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{G})} \leq c_p \varepsilon^{-N_p-p}$, for all $t \in [0,T]$ and $\varepsilon \in (0,1]$.

Note that the conditions of moderateness are natural in the sense that regularisations of distributions are moderate, namely by the structure theorems for distributions we can think that

$$\text{compactly supported distributions } \mathcal{E}'(R) \subset \{C^\infty\text{-moderate families }\}. \quad (4)$$

Thus, by (4) we see that while a solution to the Cauchy problems may not exist in the space of distributions $\mathcal{E}'(R)$, it may still exist (in a certain appropriate sense) in the space on the right hand side of (4). The moderateness assumption will be enough for our purposes. However, we note that regularisation with standard Friedrichs mollifiers is not sufficient, hence the introduction of a family $\omega(\varepsilon)$ in the above regularisations.

Let $H_s^{-\infty}$ and $H_{(s)}^{-\infty}$ be the spaces of linear continuous functionals on $\mathcal{G}_{\mathcal{R}}^s$ and $\mathcal{G}_{\mathcal{R}}^{(s)}$, respectively. Recall

$$\partial_t^2 u(t) + a(t)\mathcal{R}u(t) = 0,\ u(0) = u_0 \in L^2(G),\ \partial_t u(0) = u_1 \in L^2(G).$$

Let $s$ be a real number. We say that the net $(u_\varepsilon)_\varepsilon \subset C^\infty([0,T]; H_{\mathcal{R}}^s)$ is its very weak solution of $H^s$-type if there exist

• $C^\infty$-moderate regularisation $a_\varepsilon$ of the coefficient $a$ such that $(u_\varepsilon)_\varepsilon$ solves the following regularised problem

$$\partial_t^2 u_\varepsilon(t) + a_\varepsilon(t)\mathcal{R}u_\varepsilon(t) = 0,\ u_\varepsilon(0) = u_0 \in L^2(G),\ \partial_t u_\varepsilon(0) = u_1 \in L^2(G),$$

for all $\varepsilon \in (0,1]$, and is $C^\infty([0,T]; H_{\mathcal{R}}^s)$-moderate.

The net $(u_\varepsilon)_\varepsilon \subset C^\infty([0,T]; H_{(s)}^{-\infty})$ is a very weak solution of $H_{(s)}^{-\infty}$-type if there exist

• $C^\infty$-moderate regularisation $a_\varepsilon$ of the coefficient $a$, such that $(u_\varepsilon)_\varepsilon$ solves the regularised problem for all $\varepsilon \in (0,1]$, and is $C^\infty([0,T]; H_{(s)}^{-\infty})$-moderate.

**Theorem. Y.+Ruzhansky, JDE 2020**

Let $G$ be a graded Lie group and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree $\nu$. Let $T > 0$ and $s \in \mathbb{R}$.

• Let $a = a(t)$ be a positive distribution with compact support included in $[0, T]$, such that $a \geq a_0 > 0$. Let $u_0, u_1 \in H^s_{\mathcal{R}}$. Then there exists a very weak solution of $H^s$-type.

• Let $a = a(t)$ be a nonnegative distribution with compact support included in $[0, T]$, such that $a \geq 0$. Let $u_0, u_1 \in H^{-\infty}_{(s)}$. Then there exists a very weak solution of $H^{-\infty}_{(s)}$-type.

• The above very weak solutions are unique in the Colombeau sense.

• There is consistency (backward compatibility) in all the cases of the theorem by Ruzhansky-Taranto (when $a(t)$ is Holder or more regular). For example: Let $a(t) \geq a_0 > 0$ and $a \in C^\alpha([0, T])$ with $0 < \alpha < 1$. Let $1 \leq s < 1 + \frac{\alpha}{1-\alpha}$, $(u_0, u_1) \in H^{-\infty}_{(s)}$. Let $u$ be a very weak solution of $H^{-\infty}_{(s)}$-type. Then for any regularising families $a_\varepsilon$, any representative $(u_\varepsilon)_\varepsilon$ of $u$ converges in $C^2([0, T]; H^{-\infty}_{(s)})$ as $\varepsilon \to 0$ to the unique classical solution in $C^2([0, T]; H^{-\infty}_{(s)})$ of our Cauchy problem.

# Steps of the proof

- A regularisation of the coefficient to get a family $a_\varepsilon \in C^\infty$.
- A regularisation of data (depending on how regular it is initially).
- Once we are in the regularised (Gevrey) setting the known results imply the existence of a classical (Gevrey) solution for each $\varepsilon > 0$. However, to show that the very weak solutions exist we need to show that the dependence of solutions on the regularisation parameter $\varepsilon$ is moderate.
- To achieve this, we apply the group Fourier transform to the equation with respect to $x \in \mathbb{G}$, then using the fact that spectrum of $\pi(\mathcal{R})$ is discrete, we reduce the equation to a system. We also use the theory of the quasi-symmetrisers.
- We use families of energy inequalities depending on parameter $\varepsilon$.
- By relating parameters to the frequency, we obtain 'very weak solutions'.
- If the coefficient are regular enough we control their regularisations well. This allows to show that the difference between the classical/ultradistributonal solution and the very weak solution is negligible. This is done by an appropriate version of the energy inequality.

# Summary and conclusions

- If the coefficient of the wave equation(s) are Hölder or they are very regular (even analytic) but there are multiplicities, the class of distributions is not enough to find a solution. However, in the above cases the class of ultradistributions is enough.
- If the coefficient are continuous, $L^1_{loc}$, measures, or distributions, the class of ultradistributions is not enough. Also, the notion of a 'distributional solution' is too strong.
- Therefore, we need to extend the class of possible solutions and to further weaken the notion of a solution.
- We show: if coefficient is a distribution, a 'very weak solution' exists.
- We use explicitly given regularisations, so one gets an 'approximation' of a solution.
- If the Cauchy problem happens to have a classical, weak, or distributional solution, the 'very weak solution' recaptures it.
- if there is no distributional or ultradistributional solution, maybe we have to live with the 'very weak solution'. Having 'existence' allows further handling of solutions and their properties.

Time to talk about global well-posedness for a semilinear heat equation?

Let $\mathbb{G}$ be a connected unimodular Lie group, endowed with the Haar measure, and let $X = \{X_1, \cdots, X_k\}$ be a Hörmander system of left invariant vector fields. We consider the Cauchy problem on $\mathbb{G}$

$$\begin{cases} u_t - \mathfrak{L}u = |u|^p, \ x \in \mathbb{G}, \ t > 0, \ 1 < p < \infty \\ u(0, x) = u_0(x), x \in \mathbb{G}, \end{cases} \tag{5}$$

for $u_0 \geq 0$, where $\mathfrak{L}$ is the sub-Laplacian of $\mathbb{G}$, that is,

$$\mathfrak{L} := \sum_{i=1}^{k} X_i^2.$$

# Unimodular Lie groups

Let $\rho(x, y)$ be the Carnot-Carathéodory distance $\mathbb{G} \times \mathbb{G} \ni (x, y) \mapsto \rho(x, y) \mapsto \rho(x, y)$ associated with $X$. We denote by $\rho(x)$ the distance from the unit element of the group to $x \in \mathbb{G}$. Let $V(t)$ be the volume of the ball $B(x, t)$ centred at $x \in \mathbb{G}$ and of radius $t > 0$ for $\rho(x)$. Recall that we have $V(t) \simeq t^d$ for $t \in (0, 1)$, where $d = d(\mathbb{G}, X) \in \mathbb{N}$ is the local dimension. In the case $t \geq 1$, only two situations may occur, independently of the choice of $X$:

- either $\mathbb{G}$ has polynomial volume growth of order $D$, which means that there exists the global dimension $D = D(\mathbb{G}) \in \mathbb{N}_0$ such that $V(t) \simeq t^D, t \geq 1$,
- or $\mathbb{G}$ has exponential volume growth, that is, there exist positive constants $c_1, C_1, c_2$ and $C_2$ such that $c_1 e^{c_2 t} \leq V(t) \leq C_1 e^{C_2 t}\ t \geq 1$.

Let us also recall that the closed subgroups of nilpotent Lie groups, connected Type $R$ Lie groups, motion groups, the Mautner group and compact groups are all examples of polynomial growth groups.

# Global existence result

Let $1 < p < \infty$. Let $0 \le u_0 \in L^q(\mathbb{G})$ with $1 \le q < \infty$ and assume that

$$\int_0^\infty \|e^{-s\mathfrak{L}} u_0\|^{p-1}_{L^\infty(\mathbb{G})} ds < \frac{1}{(p-1)}. \tag{6}$$

Then there exists a non-negative continuous curve $u : [0, \infty) \to L^q(\mathbb{G})$ which is a global solution to (5) with initial value $u_0$. Moreover, we have

$$(e^{-t\mathfrak{L}} u_0)(x) \le u(t, x) \le C(e^{-t\mathfrak{L}} u_0)(x), \ \forall x \in \mathbb{G}, \ \forall t \ge 0, \tag{7}$$

for some $C > 1$ (depending on $u_0$). For example, (7) holds with

$$C = \left(1 - (p-1) \int_0^\infty \|e^{-s\mathfrak{L}} u_0\|^{p-1}_{L^\infty(\mathbb{G})} ds\right)^{-\frac{1}{p-1}}.$$

# Conclusion

When we know the behavior of the heat kernel, then one can see that to satisfy the condition (6) there appears a condition for the parameter $p$, which is usually called a Fujita exponent.

- In the case $D = 0$ (e.g. when $\mathbb{G}$ is a compact group), that is, the case when the volume growth at infinity is constant, the Cauchy problem (5) does not admit any nontrivial distributional solution $u \geq 0$ in $(0, \infty) \times \mathbb{G}$ for $1 < p < \infty$.
- In the case of polynomial volume growth, there exists a global, classical solution of (5) for $p > p_F = 1 + 2/D$ and sufficiently small non-negative $u_0$.
- When $\mathbb{G}$ has exponential volume growth, then the Cauchy problem (5) has a global, classical solution for all $1 < p < \infty$ and sufficiently small non-negative $u_0$.

**Thank you for your attention!**