On sums of squares of complex vector fields

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Outline.

In this talk I will try to give an alternate view of Kohn's result on the hypoellipticity of sums of squares of (C^∞) complex vector fields on an open set $\Omega \subset \mathbb{R}^n$. The approach is based on the strong form of Melin's inequality, exploits some symplectic invariants, that I will recall below. This will yield a new proof of Kohn's result and, furthermore, the extent to which it is stable under perturbations with lower order operators.

1. The Hörmander condition, hypoellipticity, subellipticity.
2. Kohn's Theorem and counterexamples.
3. Melin's inequality (strong form).
4. The result (case $N = 2$, $n = 3$) and some consequences.

The Hörmander condition, hypoellipticity, subellipticity.

Let $\Omega \subset \mathbb{R}^n$ be open, $X_1, \ldots, X_N \in C^\infty(\Omega; T\Omega)$ be real smooth vector fields ($X_j = X_j(x, D)$, $D = -i\partial$; actually, the iX_j are v.f.s). Let

$$P = \sum_{j=1}^{N} X_j^* X_j.$$

Let $\mathscr{L}_{\mathsf{X}}$ be the Lie algebra generated (over $C^\infty(\Omega; \mathbb{R})$) by the system $\mathsf{X} = (X_1, \ldots, X_N)$, and $\operatorname{rk} \mathscr{L}_{\mathsf{X}}(x)$ be its dimension as a real vector space at x, i.e. as $\operatorname{Span}_{\mathbb{R}}\{X_1, \ldots, X_N, [X_{j_1}, [X_{j_2}, \ldots, [X_{j_{k-1}}, X_{j_k}] \ldots]], \ 1 \leq j_h \leq N\}(x)$.
Recall: P is C^∞ hypoelliptic if $\operatorname{singsupp}(Pu) = \operatorname{singsupp}(u)$, for all $u \in \mathscr{D}'(\Omega)$ or, equivalently

$$\forall u \in \mathscr{D}'(\Omega), \ \forall V \,(\text{open}) \subset \Omega \ Pu \in C^\infty(V) \Longrightarrow u \in C^\infty(V).$$

(Analytic h.e.: replace C^∞ by C^ω.)

Hörmander's Theorem.

Suppose that at any given $x \in \Omega$ one has $\operatorname{rk} \mathscr{L}_{\mathsf{X}}(x) = n$, i.e. $\mathscr{L}_{\mathsf{X}}(x) = T_x\Omega$. Then P is C^∞ hypoelliptic.

Remark.

Lie algebra condition is not necessary (Fedii, Morimoto): one has P which is C^∞ h.e. and yet $\mathscr{L}_X(x) \subsetneq T_x\Omega$ for some x. However, when coefficients C^ω then Lie algebra condition is also necessary for C^ω h.e. (Derridj), but only necessary: existence of the Baouendi-Goulaouic operator; Treves' conjecture on analytic-hypoellipticity.

Basic estimate: the subelliptic estimate.

The main step in the proof is the following energy estimate. Suppose for simplicity that $\mathscr{L}(x)$ is spanned by the commutators up to length k (the v.f.s have length 1).

There exists $\varepsilon > 0$ (in this case $\varepsilon = 1/k$) such that $\forall K \subset\subset \Omega$ $\exists C_K$ such that

$$(SE) \qquad \|u\|_\varepsilon^2 \leq C_K\Big((Pu, u) + \|u\|_0^2\Big), \quad \forall u \in C_c^\infty(K).$$

Fundamental work related to subelliptic estimates:

Rothschild-Stein, Oleinik-Radkevich, Fefferman-Phong, Bolley-Camus-Nourrigat, Helffer-Nourrigat, Egorov, Hörmander, among others ...

Remark.

Since $0 < \varepsilon \leq 1$ ($(Pu, u) = O(\|u\|_1^2)$ by continuity), one talks also of hypoellipticity with a loss of $2 - 2\varepsilon$ derivatives.
The operator P of order m is said to be (C^∞) hypoelliptic with a loss of $r \geq 0$ at a point x_0 if $\forall u \in \mathscr{D}'(\Omega)$ and all $s \in \mathbb{R}$

$$Pu \in H^s(x_0) \Longrightarrow u \in H^{s+m-r}(x_0).$$

Remark (Hypoellipticty with a loss of many derivatives).

There are operators which are hypoelliptic with a large number of derivatives and yet they are C^∞ hypoelliptic.
As an example (more to come): let $d \geq 1$ be an integer, and let $\mu > 0$ and $\gamma \in S = \{\pm(2\ell + 1);\ \ell \in \mathbb{Z}_+\}$. Then

$$P_\gamma = (1 + x_1^{2d})(D_{x_1}^2 + \mu^2 x_1^2 D_{x_2}^2) + (\gamma + \mu x_1^{2d}) D_{x_2} - 2i\, x_1^{2d-1}(D_{x_1} + i\mu x_1 D_{x_2})$$

is still C^∞ hypoelliptic, with a loss of exactly $d + 1$ derivatives. (Theory developed by C. Parenti-P.) When $\gamma \notin S$ then P_γ is hypoelliptic with a loss of 1 derivative (Boutet De Monvel-Grigis-Helffer).

Kohn's Theorem and counterexamples.

Y. T. Siu's program to use multipliers for the $\bar\partial$-Neumann problem to get explicit construction of critical varieties that control the D'Angelo type. Let

$Z_j \in C^\infty(\Omega; \mathbb{C}T\Omega)$, $1 \le j \le N$, be smooth vector fields. Let $P = \sum_{j=1}^N Z_j^* Z_j$.

Thm. A (J. J. Kohn.)

Suppose that

$$(K) \qquad \mathrm{Span}_{\mathbb{C}}\{Z_j, [Z_j, Z_k];\ 1 \le j,k \le N\}\Big|_x = \mathbb{C}T_x\Omega, \quad \forall x \in \Omega,$$

then (SE) holds with $\varepsilon = 1/2$, that is,

$$\forall K \subset\subset \Omega\ \exists C_K > 0, \quad \|u\|^2_{1/2} \lesssim (Pu,u) + \|u\|_0^2, \quad \forall u \in C_c^\infty(K).$$

So, if $\mathscr{L}^{\mathbb{C}}_{2,Z}(x) = \mathbb{C}T_x\Omega$ for all $x \in \Omega$, then Hörmander's Thm. holds. However, *as soon as more commutators are required to generate the complexified tangent space, the result no longer holds.*

In fact:

Thm. B (J. J. Kohn.)

For any given $k \in \mathbb{Z}_+$ $\exists$ complex vector fields Z_1, Z_{2k} near $0 \in \mathbb{R}^3$ such that Z_1, Z_{2k} and their commutators of length $k+1$ span $\mathbb{C}T_0\Omega$ and when $k \geq 1$ (SE) does not hold anymore. Moreover, the operator $P = P_k$ is hypoelliptic with a loss of $k+1$ derivatives.

Consider the complex vector field in $\mathbb{R}^3_{x_1,x_2,x_3}$

$$\bar{L} := \frac{\partial}{\partial \bar{z}_1} - iz_1 \frac{\partial}{\partial x_3}, \quad z_1 = x_1 + ix_2.$$

Then $\bar{L}$ is a version of the Lewy operator

$$\bar{L}_0 = D_{x_1} + iD_{x_2} + i(x_1 + ix_2)D_{x_3}$$

which appears as the tangential CR operator on the boundary of the strictly ψ-convex domain

$$X = \{(z_1, z_2) \in \mathbb{C}^2;\ |z_1|^2 + 2\,\mathrm{Im}\, z_2 < 0\}.$$

Take than x_1, x_2, x_3, $z_1 = x_1 + ix_2$,

$$Z_1 = \bar{L} = \frac{\partial}{\partial \bar{z}_1} - iz_1 \frac{\partial}{\partial x_3}, \quad Z_{2k} = \bar{z}_1^k L = \bar{z}_1^k \frac{\partial}{\partial z} + i\bar{z}_1^{k+1} \frac{\partial}{\partial x_3},$$

$$P = P_k = Z_1^* Z_1 + Z_{2k}^* Z_{2k} = -(L\bar{L} + \bar{L}|z_1|^{2k} L).$$

Remark. Simplified examples (Christ, Parenti-P.). That of Parenti-P.:

Take $n, d \geq 1$ integers, $\mu_j > 0$, $1 \leq j \leq n$, *rationally independent*, $\gamma \in \mathbb{R}$,

$$Q(x) = \sum_{|\alpha|=d} c_\alpha x^{2\alpha}, \ x \in \mathbb{R}^n, \ \sum_{|\alpha|=d} c_\alpha > 0, \quad X_j = D_{x_j} - i\mu_j x_j D_y, \quad 1 \leq j \leq n,$$

$$P = \sum_{j=1}^n X_j^* X_j + \sum_{j=1}^n X_j(Q(x) X_j^*) + (\gamma + |\mu|) D_y, \ y \in \mathbb{R}.$$

P is h.e. with a loss of exactly 1 derivative iff (B-G-H)

$$\gamma \notin S := \{\pm(2\langle \ell, \mu \rangle + |\mu|); \ \ell \in \mathbb{Z}_+\},$$

when $\gamma \in S$ then P is h.e. with a loss of exactly $d+1$ derivatives. (Parenti-P.)

Melin's inequality (strong form).

Thm. (A. Melin.)

Let $P = P^*$ be an mth-order (ψ)do on $\Omega \subset \mathbb{R}^n$. Suppose that $p_m(x,\xi) \geq 0$ for all $(x,\xi) \in T^*\Omega \setminus 0$ and that

$$(sM) \qquad p_m(x,\xi) = 0 \Longrightarrow p^s_{m-1}(x,\xi) + \mathrm{Tr}^+ F(x,\xi) > 0.$$

Then for all compact $K \subset \Omega$ $\exists c_K, C_K > 0$ such that

$$(ME) \qquad (Pu,u) \geq c_K \|u\|^2_{(m-1)/2} - C_K \|u\|^2_{(m-2)/2}, \quad \forall u \in C^\infty_c(K).$$

When P is a pdo then m even

Σ is symmetric under $(x,\xi) \longmapsto (x,-\xi)$ and (sM) is equivalent to

$$|p^s_{m-1}(\rho)| < \mathrm{Tr}^+ F(\rho).$$

Melin's condition is described in terms of symplectic invariants.

Symplectic invariants. $p_m \geq 0$ **yields** ($\Sigma = p_m^{-1}(0) \subset T^*\Omega \setminus 0$)**:**

- the **subprincipal symbol** (invariant at zeros of order 2 of the principal symbol; pos. homog. deg. $m-1$)

$$p^s_{m-1}(x,\xi) = p_{m-1}(x,\xi) + \frac{i}{2}\sum_{j=1}^{n} \partial^2_{x_j\xi_j} p_m(x,\xi);$$

- the **fundamental matrix** $F(\rho)$, $\rho \in \Sigma$ (also **Hamilton map**; linearization of $\exp(tH_{p_m})$ at Σ), Hessian of p_m is invariant on Σ,

$$\sigma(v, F(\rho)w) = \frac{1}{2}\langle \mathrm{Hess}(p_m)(\rho)v, w\rangle, \quad v, w \in T_\rho T^*\Omega;$$

- F is skew-symmetric with respect to σ.

Symplectic invariants (cont'd): $\rho \in \Sigma$

- Spectral structure: $\operatorname{Ker} F(\rho) \subset \operatorname{Ker} F(\rho)^2 = \operatorname{Ker} F(\rho)^3$,

$$\operatorname{Spec}(F(\rho)) = \{0\} \cup \underbrace{\{\pm i\mu_j;\ \mu_j > 0,\ 1 \leq j \leq \nu\}}_{\text{regular eigenvalues}},$$

$$T_\rho T^*\Omega = \operatorname{Ker} F(\rho)^2 \oplus \operatorname{Im} F(\rho)^2;$$

- The **positive trace** of $F(\rho)$

$$\operatorname{Tr}^+ F(\rho) = \sum_{\mu > 0,\, i\mu \in \operatorname{Spec}(F(\rho))} \mu$$

(positively homogeneous of degree $m - 1$).

Question: Is there a link between Kohn's result and Melin's inequality?

- The point is to understand the symplectic content of condition (K) and understand its relation (if any) with condition (sM).
- Advantages: propagation of smoothness (along Riemann surfaces in $T^*\Omega$) and unique continuation.

Our setting:

$Z_j(x,\xi) = \langle \zeta_j(x), \xi\rangle$, the symbol (moment map) of the differential operator associated with the complex vector field $\zeta_j = \alpha_{2j-1} + i\alpha_{2j} \in C^\infty(\Omega; \mathbb{C}T\Omega)$, $1 \leq j \leq N$ (with no common critical points). Hence $Z_j(x,\xi) = X_{2j-1}(x,\xi) + iX_{2j}(x,\xi)$. Convenient to work with

$$P = \sum_{j=1}^{N} Z_j^{\mathrm{w}}(x,D) Z_j^{\mathrm{w}}(x,D),$$

where Z_j^{w} stands for the Weyl quantizazion of $Z_j(x,\xi)$. Hence

$$Z_j^{\mathrm{w}}(x,D) = \tilde{Z}_j(x,D), \quad \tilde{Z}_j(x,\xi) = Z_j(x,\xi) \underbrace{- \frac{i}{2} \sum_{k=1}^{n} \frac{\partial \zeta_j}{\partial x_k}(x)}_{= -\frac{i}{2}\operatorname{div} Z_j = \operatorname{sub}(Z_j)}.$$

Remark:

P (written in terms of Z_j^{w}) satisfies (SE) iff $\sum_{j=1}^{N} \tilde{Z}_j(x,D)^* \tilde{Z}_j(x,D)$ satisfies (SE) with the **same** ε.

The characteristic set is in this case

$$\Sigma = \bigcap_{j=1}^{N} Z_j^{-1}(0).$$

On $T^*\Omega \setminus \Sigma$ we thus have ellipticity. We set

$$W(x) = \mathrm{Span}_{\mathbb{R}}\{\alpha_{2j-1}(x),\ \alpha_{2j}(x);\ \leq j \leq N\}$$

$$\Sigma = \{(x,\xi) \in T^*\Omega;\ 0 \neq \xi \in W(x)^{\perp}\},$$

$$W_{\mathbb{C}}(x) = W_0(x) \oplus W_1(x),$$

$$W_0(x) = \mathrm{Span}_{\mathbb{C}}\{\zeta_j(x);\ 1 \leq j \leq N\},$$

$$W_1(x) = \mathrm{Span}_{\mathbb{C}}\{[\zeta_j,\zeta_k](x);\ 1 \leq j,k \leq N\,,\ [\zeta_j,\zeta_k](x) \notin W_0(x)\},$$

where, depending on x, we may have $W_1(x) = \{0\}$. Of course, important points: $x \in \pi(\Sigma)$, $\pi\colon\ T^*\Omega \longrightarrow \Omega$ being the canonical projection.

$$(K) \iff W_{\mathbb{C}}(x) = \mathbb{C}T_x\Omega,\ \forall x \in \pi(\Sigma).$$

Invariants of the operator P:

- The principal symbol:

$$p_2(x,\xi) = \sum_{j=1}^{N} |Z_j(x,\xi)|^2;$$

- The subprincipal symbol:

$$p_1^s(x,\xi) = -\frac{i}{2}\sum_{j=1}^{N}\{\bar{Z}_j, Z_j\}(x,\xi) = \sum_{j=1}^{N}\langle[\alpha_{2j-1},\alpha_{2j}](x),\xi\rangle = \sum_{j=1}^{N}\{X_{2j-1}, X_{2j}\}(x,\xi);$$

- The Hamilton map at $\rho \in \Sigma$: with H_j the Hamilton vector field of X_j,

$$H_j = \sum_{k=1}^{n}\Big(\frac{\partial X_j}{\partial \xi_k}\frac{\partial}{\partial x_k} - \frac{\partial X_j}{\partial x_k}\frac{\partial}{\partial \xi_k}\Big),$$

$$F(\rho)w = \sum_{j=1}^{N}\Big(\sigma(w, H_{2j-1}(\rho)H_{2j-1}(\rho) + \sigma(w, H_{2j}(\rho))H_{2j}(\rho)\Big),\ w \in T_\rho T^*\Omega.$$

A meaningful case, $n = 3$, $N = 2$.

Crucial observation:

(K) holds at x_0 iff for any given $0 \neq \xi \in W(x_0)^\perp$ either

$$\{X_1, X_3\}(x_0, \xi) - \{X_2, X_4\}(x_0, \xi) \neq 0,$$

or

$$\{X_1, X_4\}(x_0, \xi) + \{X_2, X_3\}(x_0, \xi) \neq 0.$$

Problem:

Check whether Melin's condition is fulfilled. We need to study $\mathrm{Spec}(F(\rho))$.

$\mathrm{Tr}^+F(\rho)$ and (sM):

One studies $F(\rho)w = i\mu w$, $\mu > 0$, $w \in \mathbb{C}\mathrm{Im}(F(\rho)^2)$. With

$$s = \sum_{1\le j<k\le 4} \{X_j, X_k\}^2,$$

$$t = \{X_1, X_2\}\{X_3, X_4\} + \{X_1, X_4\}\{X_2, X_3\} - \{X_1, X_3\}\{X_2, X_4\},$$

one has

$$\mathrm{Tr}^+F = \sqrt{s + 2|t|}$$

whence, recalling that $p_1^s = \{X_1, X_2\} + \{X_3, X_4\}$, (sM) is the following condition:

$$s + 2|t| > |p_1^s|^2.$$

End of computation:

This is granted if

$$\Big(\{X_1, X_2\} + \{X_3, X_4\}\Big)^2 + \Big(\{X_1, X_3\} - \{X_2, X_4\}\Big)^2 + 2\{X_1, X_2\}\{X_3, X_4\} > 2\{X_1, X_2\}\{X_3, X_4\},$$

which holds if Kohn's condition is satisfied.

Thm. (P.; $n = 3$, $N = 2$)

Condition (K) yields condition (sM).

The Thm. yields the following perturbation result.

Cor. (P.; $n = 3$, $N = 2$)

Suppose condition (K). Define for $\rho \in \Sigma$, $\kappa = \kappa(\rho) > 0$ by

$$\kappa^2 := \Big(\{X_1, X_2\} + \{X_3, X_4\}\Big)^2 + \Big(\{X_1, X_4\} + \{X_2, X_3\}\Big)^2 + \Big(\{X_1, X_3\} - \{X_2, X_4\}\Big)^2,$$

$$\Lambda_\pm(\rho) := -\Big(\{X_1, X_2\} + \{X_3, X_4\}\Big) \pm \kappa.$$

Then $P + Z_0$, with Z_0 1st-order having real symbol, satisfies (SE) with $\varepsilon = 1/2$ provided that for all $\rho \in \Sigma$

$$\Lambda_-(\rho) < Z_0(\rho) < \Lambda_+(\rho).$$

In particular, when Z_0 is differential, we must have

$$|Z_0(\rho)| < \min\{-\Lambda_-(\rho), \Lambda_+(\rho)\} = \kappa(\rho) - |\{X_1, X_2\}(\rho) + \{X_3, X_4\}(\rho)|.$$

"More generally we can ask for properties of P or rather its characteristic polynomial which are intrinsic in the sense that they are more or less equivalent to properties of the solutions. Questions of this nature have no physical background but a very solid motivation: mathematical curiosity. They lead Hadamard to the fruitful notion of correctly set boundary problems.

L. Gårding: *Hörmander's work on partial differential operators*. ICM 1962