Global dynamics for semilinear heat equations in energy spaces associated with self-adjoint operators

Koichi Taniguchi (Nagoya University)

Joint work with Masahiro Ikeda (RIKEN/Keio University)

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Let $d \geq 3$. We consider the Cauchy problem for semilinear heat equation:

$$\text{(P)} \qquad \begin{cases} \partial_t u - \Delta u = |u|^{p^*-1}u & \text{in } (0,T) \times \mathbb{R}^d, \\ u(0) = u_0 \in \dot{H}^1(\mathbb{R}^d), \end{cases}$$

where $T > 0$ and $p^* := \frac{d+2}{d-2}$ is the $\dot{H}^1$-critical exponent.

Sobolev inequality:

$$\|f\|_{L^{p^*+1}(\mathbb{R}^d)} \leq S_{p^*+1} \|f\|_{\dot{H}^1(\mathbb{R}^d)}, \quad f \in \dot{H}^1(\mathbb{R}^d),$$

where S_{p^*+1} is the best constant.

Energy: The energy functional $E : \dot{H}^1(\mathbb{R}^d) \to \mathbb{R}$ is defined by

$$E(u) := \frac{1}{2}\|u\|^2_{\dot{H}^1(\mathbb{R}^d)} - \frac{1}{p+1}\|u\|^{p^*+1}_{L^{p^*+1}(\mathbb{R}^d)},$$

and the energy is (formally) dissipated along solutions to (P):

$$\frac{d}{dt}E(u(t)) = -\int_{\mathbb{R}^d} |u_t(t)|^2 \, dx \leq 0.$$

Ground state solution:

$$W(x) := \frac{1}{\left(1 + \frac{|x|^2}{d(d-2)}\right)^{\frac{d-2}{2}}} \in \dot{H}^1(\mathbb{R}^d)$$

is a ground state solution to the corresponding stationary problem $-\Delta u = |u|^{p^*-1}u$, $u \in \dot{H}^1(\mathbb{R}^d)$.

Classification of behavior of solutions

Let u be a solution to (P) with $u(0) = u_0 \in \dot{H}^1(\mathbb{R}^d)$. Suppose $E(u_0) < E(W)$. Then

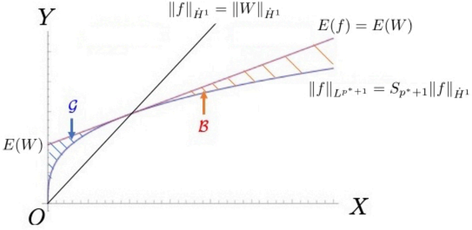
(i) $\|u_0\|_{\dot{H}^1(\mathbb{R}^d)} < \|W\|_{\dot{H}^1(\mathbb{R}^d)} \Rightarrow u$ is global and dissipative, i.e., $\lim_{t\to\infty} \|u(t)\|_{\dot{H}^1(\Omega)} = 0$.

(ii) $\|u_0\|_{\dot{H}^1(\mathbb{R}^d)} > \|W\|_{\dot{H}^1(\mathbb{R}^d)} \Rightarrow u$ blows up in finite time.

$\mathcal{G} := \{f \in \dot{H}^1 : E(u_0) < E(W), \|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}\} \cup \{0\}$,

$\mathcal{B} := \{f \in \dot{H}^1 : E(u_0) < E(W), \|u_0\|_{\dot{H}^1} > \|W\|_{\dot{H}^1}\}$.

$$X = \frac{1}{p^*+1}\|f\|_{L^{p^*+1}}^{p^*+1}, \quad Y = \frac{1}{2}\|f\|_{\dot{H}^1}^2$$



Known results

There are many results on the global behavior and blow up in finite time in the following cases:

- The Cauchy problem for semilinear heat equation on $\mathbb{R}^d$;
- The Dirichlet problem for semilinear heat equation on a bounded domain Ω.

Energy subcritical case ($1 < p < p^*$):

- Below the ground state
  $\cdots$ D. H. Sattinger (1968), M. Tsutsumi (1972), L. E. Payne-D. H. Sattinger (1975), R. Ikehata-T. Suzuki (1996), F. Gazzola-T. Weth (2005), etc.
- Related results
  $\cdots$ F. Gazzola-T. Weth (2005), F. Dickstein-N. Mizoguchi-P. Souplet-F. Weissler (2011).

Energy critical case: $p = p^*$

- Below the ground state $\cdots$ S. Gustafson-D. Roxanas (2018): $\Omega = \mathbb{R}^4$.
  But they impose the additional assumption $u_0 \in L^2(\mathbb{R}^4)$
  in the blow up result.

  (Schrödinger, wave equations: C. Kenig-F. Merle (2006, 2008), R. Killip-M. Visan (2010)).
- Related results $\cdots$ R. Schweyer (2012), C. Collot-F. Merle-P. Raphaël (2017).

Our aim

Aim.

The aim is to generalize the above results to more general situations:

$$\partial_t u + \mathcal{L}u = |u|^{p-1}u$$

with a self-adjoint operator $\mathcal{L}$ such as the Dirichlet Laplacian on an open set, Robin Laplacian on an exterior domain, and Schrödinger operators, etc.

Let $d \geq 1$ and Ω be an open set in $\mathbb{R}^d$, and let $\mathcal{L}$ be a suitable self-adjoint operator on $L^2(\Omega)$.

Problem.

We consider

$$(\mathrm{P}_{\mathcal{L}}) \qquad \begin{cases} \partial_t u + \mathcal{L}u = F(u) & \text{in } (0,T) \times \Omega, \\ u(0) = u_0 \in \mathcal{E}(\mathcal{L}), \end{cases}$$

where $T > 0$, and $F(u)$ and $\mathcal{E}(\mathcal{L})$ are one of the following:

(a) (Subcritical) $F(u) = -u + |u|^{p-1}u$ $(1 < p < p^*)$ and $\mathcal{E}(\mathcal{L}) = H^1(\mathcal{L})$;

(b) (Critical) $F(u) = |u|^{p^*-1}u$ $(d \geq 3)$ and $\mathcal{E}(\mathcal{L}) = \dot{H}^1(\mathcal{L})$.

- **(Sobolev spaces)** The norms of $H^1(\mathcal{L})$ and $\dot{H}^1(\mathcal{L})$ are given as follows:

$$\|f\|_{H^1(\mathcal{L})} := \|(I+\mathcal{L})^{\frac{1}{2}} f\|_{L^2(\Omega)}, \quad \|f\|_{\dot{H}^1(\mathcal{L})} := \|\mathcal{L}^{\frac{1}{2}} f\|_{L^2(\Omega)},$$

  where I is the identity operator on $L^2(\Omega)$.

- **(Energy functional)** $E_{\mathcal{L}}(u) := \frac{1}{2}\|u\|^2_{\mathcal{E}(\mathcal{L})} - \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\Omega)}$.

- **(Nehari functional)** $J_{\mathcal{L}}(f) := \|f\|_{\mathcal{E}(\mathcal{L})}^2 - \|f\|_{L^{p+1}(\Omega)}^{p+1}$.
- **(Critical value of $E_{\mathcal{L}}$)** $l_{\mathcal{L}} := \inf\{E_{\mathcal{L}}(f) : f \in \mathcal{E}(\mathcal{L}) \setminus \{0\},\ J_{\mathcal{L}}(f) = 0\}$.

Remark. In the critical case when $\mathcal{L} = -\Delta$ on $\mathbb{R}^d$,

- $J_{-\Delta}(u_0) > 0 \iff \|u_0\|_{\dot{H}^1(\mathbb{R}^d)} < \|W\|_{\dot{H}^1(\mathbb{R}^d)}$,
  $J_{-\Delta}(u_0) < 0 \iff \|u_0\|_{\dot{H}^1(\mathbb{R}^d)} > \|W\|_{\dot{H}^1(\mathbb{R}^d)}$.
- $l_{-\Delta} = E(W)$.

Therefore, in this case, the classification result is equivalent to the following:

Let u be a solution to (P$_{\mathcal{L}}$) with $u(0) = u_0 \in \mathcal{E}(\mathcal{L})$. Suppose $E_{\mathcal{L}}(u_0) < l_{\mathcal{L}}$. Then

(i) $J_{\mathcal{L}}(u_0) > 0 \Rightarrow u$ is global and dissipative, i.e., $\lim_{t\to\infty} \|u(t)\|_{\mathcal{E}(\mathcal{L})} = 0$.

(ii) $J_{\mathcal{L}}(u_0) < 0 \Rightarrow u$ blows up in finite time.

Energy subcritical case (a): $1 < p < p^*$

Assumption (A). For any $2 \le q < p^* + 1$, there exist $C > 0$ and $\omega < 1$ such that

$$\|e^{-t\mathcal{L}}\|_{L^2(\Omega)\to L^q(\Omega)} \le Ct^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{q})}e^{\omega t}, \quad t > 0.$$

Remark. Assumption (A) $\Rightarrow H^1(\mathcal{L}) \hookrightarrow L^{p+1}(\Omega)$ $(1 \le p < p^*)$.

Theorem 1 (Ikeda-T., arXiv:1902.01016).

The problem ($\mathrm{P}_{\mathcal{L}}$) is local well-posedness in $H^1(\mathcal{L})$. Moreover, if $u_0 \in H^1(\mathcal{L})$ satisfies $E_{\mathcal{L}}(u_0) \le l_{\mathcal{L}}$, then:

(i) $J_{\mathcal{L}}(u_0) > 0 \Rightarrow$ The solution u is global and $\lim_{t\to\infty} \|u(t)\|_{H^1(\mathcal{L})} = 0$.

(ii) $J_{\mathcal{L}}(u_0) < 0 \Rightarrow$ The solution u blows up in finite time.

Energy critical case (b): $p = p^*$

Assumption (B). Let $K_{\mathcal{L}}(t;x,y)$ be an integral kernel of the semigroup $\{e^{-t\mathcal{L}}\}_{t>0}$. There exist $C_1, C_2 > 0$ such that:

$$|K_{\mathcal{L}}(t;x,y)| \le C_1 t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{C_2 t}}, \quad t > 0, \text{ a.e. } x, y \in \Omega.$$

Remark. Assumption (B) $\Rightarrow \dot{H}^1(\mathcal{L}) \hookrightarrow L^{p^*+1}(\Omega)$ $(d \ge 3)$.

Lemma 1.

Suppose that L satisfies Assumption (B). Then the following assertions hold:

(i) For any $1 \le q \le r \le \infty$,

(1)
$$\|e^{-tL}\|_{L^q(\Omega)\to L^r(\Omega)} \le C t^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{r})}, \quad t > 0.$$

(ii) Let $1 < q \le r \le \infty$ and $\frac{1}{\gamma} = \frac{d}{2}(\frac{1}{q} - \frac{1}{r})$. Then

(2)
$$\|e^{-tL} f\|_{L^\gamma(\mathbb{R}_+; L^r(\Omega))} \le C\|f\|_{L^q(\Omega)}, \quad f \in L^q(\Omega),$$

(3)
$$\|e^{-tL} f\|_{L^2(\mathbb{R}_+; \dot{H}^1(L))} \le C\|f\|_{L^2(\Omega)}, \quad f \in L^2(\Omega).$$

Energy critical case (b): $p = p^*$

Lemma 1.

(iii) Let $1 \le q_1 \le q_2 \le \infty$ and $1 < \gamma_1, \gamma_2 < \infty$ satisfying

$$\frac{1}{\gamma_2} = \frac{1}{\gamma_1} + \frac{d}{2}\left(\frac{1}{q_1} - \frac{1}{q_2}\right) - 1, \quad \frac{d}{2}\left(\frac{1}{q_1} - \frac{1}{q_2}\right) < 1.$$

Then

(4) $$\left\| \int_0^t e^{-(t-s)L} F(s)\, ds \right\|_{L^{\gamma_2}(\mathbb{R}_+; L^{q_2}(\Omega))} \le C \|F\|_{L^{\gamma_1}(\mathbb{R}_+; L^{q_1}(\Omega))},$$

(5) $$\left\| \int_0^t e^{-(t-s)L} F(s)\, ds \right\|_{L^{\infty}(\mathbb{R}_+; \dot{H}^1(L))} \le C \|F\|_{L^{2}(\mathbb{R}_+; L^{2}(\Omega))}.$$

Remark. The space-time estimates (3) and (5) are better than endpoint estimates appeared in the case of Schrödinger equation. These estimates (3) and (5) are useful in the proof of LWP in $\dot{H}^1(\mathcal{L})$, because (3) and (5) allow us to avoid to estimate $\|\mathcal{L}^{\frac{1}{2}}(|u|^{p-1}u)\|$.

Energy critical case (b): $p = p^*$

Theorem 2 (Ikeda-T., arXiv:1902.01016).

The problem (P$_{\mathcal{L}}$) is local well-posedness in $\dot{H}^1(\mathcal{L})$. Moreover, if $u_0 \in \dot{H}^1(\mathcal{L})$ satisfies $E_{\mathcal{L}}(u_0) \leq l_{\mathcal{L}}$, then:

(i) $J_{\mathcal{L}}(u_0) > 0$ and $\|u_0\|_{\dot{H}^1(L)} \ll 1 \Rightarrow$ The solution u is global and $\lim_{t\to\infty} \|u(t)\|_{\dot{H}^1(\mathcal{L})} = 0$.

(ii) $J_{\mathcal{L}}(u_0) < 0 \Rightarrow$ The solution u blows up in finite time.

Remark.

- When $\mathcal{L} = -\Delta$ on $\mathbb{R}^4$, it is known that the smallness assumption can be removed via the profile decomposition plus backward uniqueness (see S. Gustafson-D. Roxanas (2018)).
- We succeed in removing the additional assumption $u_0 \in L^2(\mathbb{R}^4)$ in the blow up result of Gustafson-Roxanas (2018) by modifications of the blow up argument in Gustafson-Roxanas (2018) and the cut-off argument.

Examples of $\mathcal{L}$

- $\mathcal{L}$ is the Dirichlet Laplacian $-\Delta_D$ on an open set
  $\Rightarrow$ $\mathcal{L}$ satisfies Assumptions (A) and (B) (see E. M. Ouhabaz (2005)).
- $\mathcal{L}$ is the Robin Laplacian on an exterior domain with compact and Lipschitz boundary
  $\Rightarrow$ $\mathcal{L}$ satisfies (A) and (B) (see H. Kovařík-D. Mugnolo (2018)).
- $\mathcal{L}$ is the Schrödinger operator $-\Delta_D + V$ on an open set with a Kato potential $V = V(x)$ satisfying the smallness condition on its negative part
  $\Rightarrow$ $\mathcal{L}$ satisfies (A) and (B) (see P. D'Ancona-V. Pierfelice (2005), T. Iwabuchi-T. Matsuyama-T. (2018)).
- $\mathcal{L}$ is the Schrödinger operator $-\Delta - c/|x|^2$ with $0 < c \le (d-2)^2/4$
  $\Rightarrow$ $\mathcal{L}$ satisfies (A), and not (B) (see N. Ioku-G. Metafune-M. Sobajima-C. Spina (2016), N. Ioku-T. Ogawa (2019)).

Variational estimates

Define

$$\mathcal{G} := \{f \in \mathcal{E}(\mathcal{L}) : E_{\mathcal{L}}(f) < l_{\mathcal{L}}, J_{\mathcal{L}}(f) > 0\} \cap \{0\},$$

$$\mathcal{B} := \{f \in \mathcal{E}(\mathcal{L}) : E_{\mathcal{L}}(f) < l_{\mathcal{L}}, J_{\mathcal{L}}(f) < 0\}.$$

Then we have the following:

Lemma 2.

(i) If $u_0 \in \mathcal{G}$ ($\mathcal{B}$ resp.), then $u(t) \in \mathcal{G}$ ($\mathcal{B}$ resp.) for any $t \in [0, T_m)$.

(ii) If $u_0 \in \mathcal{G}$, then there exists $\delta > 0$ such that

$$J_{\mathcal{L}}(u(t)) \geq \delta \|u(t)\|^2_{\mathcal{E}(\mathcal{L})}, \quad t \in [0, T_m).$$

(iii) If $u_0 \in \mathcal{B}$, then there exists $\delta > 0$ such that

$$J_{\mathcal{L}}(u(t)) \leq -(p+1)\{l_{\mathcal{L}} - E_{\mathcal{L}}(u(t))\}, \quad t \in [0, T_m).$$

The proof is almost the same as in Kenig-Merle (2006, 2008).

We consider only the assertion (ii), i.e.,

(6) $$J_{\mathcal{L}}(u_0) < 0 \Rightarrow \text{the solution } u \text{ blows up in finite time.}$$

Let $u_0 \in \mathcal{B}$. Suppose $T_m = T_m(u_0) = +\infty$. We take $\chi_R \in C_0^\infty(\mathbb{R}^n)$ such that

$$\chi_R(x) = \begin{cases} 1, & |x| \le R, \\ 0, & |x| \ge R+1 \end{cases}$$

and define

$$I_R(t) := \int_0^t \|\chi_R u(s)\|_{L^2(\Omega)}^2 \, ds + A, \quad t \ge 0,$$

where $R > 0$ and $A > 0$, which are chosen later. Here we note that

$$\chi_R u(t) \in L^2(\Omega), \quad t > 0,$$

since

(7) $$\|\chi_R u(t)\|_{L^2(\Omega)} \le C_R \|u(t)\|_{L^{p^*+1}(\Omega)} \le C_R' \|u(t)\|_{\dot{H}^1(\mathcal{L})},$$

by Hölder's inequality and the Sobolev embedding $\dot{H}^1(\mathcal{L}) \hookrightarrow L^{p^*+1}(\Omega)$.

It suffices to show that

$$I_R''(t)I_R(t) - (1+\alpha)I_R'(t)^2 > 0, \quad t > 0 \tag{8}$$

for some $\alpha > 0$ and $R > 0$. In fact,

$$\begin{aligned}
I_R''(t)I_R(t) - (1+\alpha)I_R'(t)^2 > 0 \iff & \frac{d}{dt}\left(\frac{I_R'(t)}{I_R(t)^{\alpha+1}}\right) > 0 \\
\implies & \frac{I_R'(t)}{I_R(t)^{\alpha+1}} > \frac{I'(0)}{I(0)^{\alpha+1}} = \frac{\|\chi_R u_0\|_{L^2(\Omega)}^2}{A^{\alpha+1}} =: a \\
\implies & \frac{1}{\alpha}\left(\frac{1}{I(0)^\alpha} - \frac{1}{I_R(t)^\alpha}\right) > at \\
\implies & I_R(t)^\alpha > \frac{I(0)^\alpha}{1 - I(0)^\alpha \alpha a t} \to +\infty
\end{aligned}$$

as $t \to 1/(I_R(0)^\alpha \alpha a) = A/(\alpha \|\chi_R u_0\|_{L^2(\Omega)}^2) =: \tilde{t}$ $(< +\infty)$. Hence

$$\limsup_{t \to \tilde{t}-} \|\chi_R u(t)\|_{L^2(\Omega)} = +\infty.$$

Therefore we find from (7) that

$$\limsup_{t \to \tilde{t}-} \|u(t)\|_{\dot{H}^1(L)} = +\infty. \tag{9}$$

On the other hand, we have $u \in C([0,T]; \dot{H}^1(L))$ for any $T > 0$ by the assumption $T_m = +\infty$. This contradicts (9). Thus we conclude (6).

§3 Outline of proof of Theorem 2 (ii)

We show (8), i.e.,

$$I''_R(t)I_R(t) - (1+\alpha)I'_R(t)^2 > 0, \quad t > 0$$

for some $R > 0$ and $\alpha > 0$. Now

$$I'_R(t) = \|\chi_R u(t)\|^2_{L^2(\Omega)},$$

$$\begin{aligned} I''_R(t) &= 2(\chi_R u(t), \chi_R u_t(t))_{L^2(\Omega)} \\ &= 2(u(t), u_t(t))_{L^2(\Omega)} + 2((\chi_R^2 - 1)u(t), u_t(t))_{L^2(\Omega)} \\ &= -2J_{\mathcal{L}}(u(t)) + 2((\chi_R^2 - 1)u(t), u_t(t))_{L^2(\Omega)}. \end{aligned}$$

Here we note that

(10) $$\sup_{t\in(0,T)} |(u(t), u_t(t))_{L^2(\Omega)}| < \infty,$$

since $u \in L^\infty([0,T); \dot{H}^1(\mathcal{L}))$ and $u_t \in L^\infty([0,T); \dot{H}^{-1}(\mathcal{L}))$ by Lemma 1.

$$
\begin{aligned}
I_R'(t)^2 &= \left(\|\chi_R u_0\|_{L^2(\Omega)}^2 + 2\int_0^t (\chi_R u, \chi_R u_t)_{L^2(\Omega)}\, ds \right)^2 \\
&\le (1+\varepsilon^{-1})\|\chi_R u_0\|_{L^2(\Omega)}^4 + 4(1+\varepsilon)\left(\int_0^t (\chi_R u, \chi_R u_t)_{L^2(\Omega)}\, ds \right)^2 \\
&\le (1+\varepsilon^{-1})\|u_0\|_{L^2(\Omega)}^4 + 4(1+\varepsilon)\left(\int_0^t \|\chi_R u(s)\|_{L^2(\Omega)}^2\, ds \right)\left(\int_0^t \|\chi_R u_t(s)\|_{L^2(\Omega)}^2\, ds \right)
\end{aligned}
$$

for any $\varepsilon > 0$, and estimate $I_R''(t)$ from below as

$$
\begin{aligned}
I_R''(t) &\ge 2(p^*+1)\{l_L - E_L(u(t))\} + 2((\chi_R^2 - 1)u(t), u_t(t))_{L^2(\Omega)} \\
&\ge 2(p^*+1)\left(l_L - E_L(u_0) + \int_0^t \|u_t(s)\|_{L^2(\Omega)}^2\, ds \right) + 2((\chi_R^2 - 1)u(t), u_t(t))_{L^2(\Omega)} \\
&\ge 2(p^*+1)\left(l_L - E_L(u_0) + \int_0^t \|\chi_R u_t(s)\|_{L^2(\Omega)}^2\, ds \right) + 2((\chi_R^2 - 1)u(t), u_t(t))_{L^2(\Omega)}
\end{aligned}
$$

by Lemma 2 (iii) and the energy identity

$$
E_L(u(t)) = E_L(u_0) - \int_0^t \|u_t(s)\|_{L^2(\Omega)}^2\, ds
$$

Here we note that $u_t \in L^2((0,T); L^2(\Omega))$ by Lemma 1.

Let $\alpha > 0$. By summarizing the above estimates, we have

$$\begin{aligned}
&I_R''(t)I_R(t) - (1+\alpha)I_R'(t)^2 \\
&\geq 2(p^*+1)\left(l_L - E_L(u_0) + \int_0^t \|\chi_R u_t(s)\|_{L^2(\Omega)}^2\,ds\right)\left(\int_0^t \|\chi_R u(s)\|_{L^2(\Omega)}^2\,ds + A\right) \\
&- 4(1+\alpha)(1+\varepsilon)\left(\int_0^t \|\chi_R u(s)\|_{L^2(\Omega)}^2\,ds\right)\left(\int_0^t \|\chi_R u_t(s)\|_{L^2(\Omega)}^2\,ds\right) \\
&- (1+\alpha)(1+\varepsilon^{-1})\|u_0\|_{L^2(\Omega)}^4 - 2|((\chi_R^2-1)u(t), u_t(t))_{L^2(\Omega)}|.
\end{aligned}$$

Here it is ensured by (10) that

$$|((\chi_R^2-1)u(t), u_t(t))_{L^2(\Omega)}| \to 0 \quad \text{as } R \to +\infty.$$

Hence, taking α, ε sufficiently small and A sufficiently large such that

$$2(p^*+1) > 4(1+\alpha)(1+\varepsilon), \quad 2(p^*+1)\{l_L - E_L(u_0)\}A > (1+\alpha)(1+\varepsilon^{-1})\|u_0\|_{L^2(\Omega)}^4,$$

and R sufficiently large, we obtain (8). □

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$$
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\end{aligned}
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Here it is ensured by (10) that

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Hence, taking α, ε sufficiently small and A sufficiently large such that

$$2(p^*+1) > 4(1+\alpha)(1+\varepsilon), \quad 2(p^*+1)\{l_L - E_L(u_0)\}A > (1+\alpha)(1+\varepsilon^{-1})\|u_0\|_{L^2(\Omega)}^4,$$

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$$\begin{aligned}
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&- 4(1+\alpha)(1+\varepsilon)\left(\int_0^t \|\chi_R u(s)\|_{L^2(\Omega)}^2\,ds\right)\left(\int_0^t \|\chi_R u_t(s)\|_{L^2(\Omega)}^2\,ds\right) \\
&- (1+\alpha)(1+\varepsilon^{-1})\|u_0\|_{L^2(\Omega)}^4 - 2|((\chi_R^2 - 1)u(t), u_t(t))_{L^2(\Omega)}|.
\end{aligned}$$

Here it is ensured by (10) that

$$|((\chi_R^2 - 1)u(t), u_t(t))_{L^2(\Omega)}| \to 0 \quad \text{as } R \to +\infty.$$

Hence, taking α, ε sufficiently small and A sufficiently large such that

$$2(p^*+1) > 4(1+\alpha)(1+\varepsilon), \quad 2(p^*+1)\{l_L - E_L(u_0)\}A > (1+\alpha)(1+\varepsilon^{-1})\|u_0\|_{L^2(\Omega)}^4,$$

and R sufficiently large, we obtain (8). $\square$

Thank you for your attention.