A Fourier integral operator approach to the sub-Riemannian wave equation

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joint work with

Detlef Müller and Sebastiano Nicolussi Golo

Dispersive and subelliptic PDEs
Pisa, 11 February 2020

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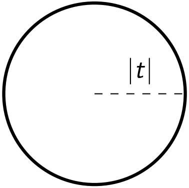

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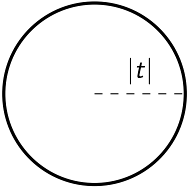

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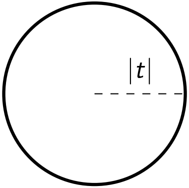

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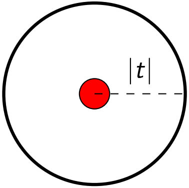

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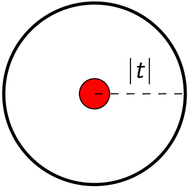

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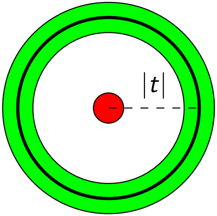

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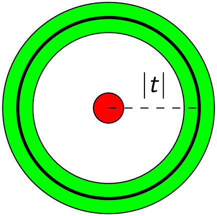

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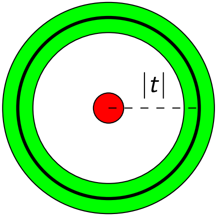

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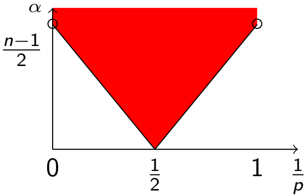

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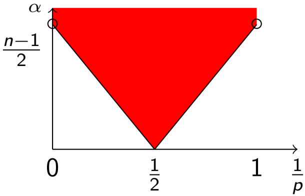

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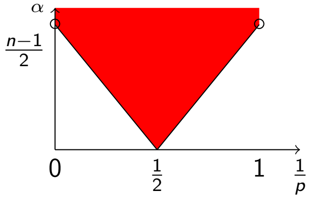

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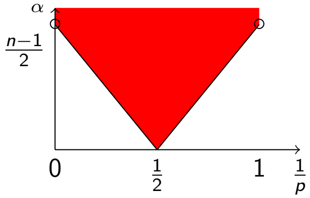

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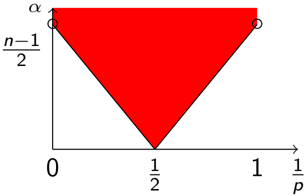

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Corollary of Mihlin–Hörmander multiplier theorem

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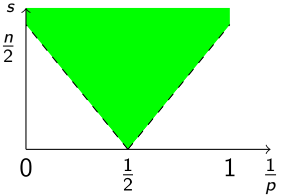

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        - $H_x^\ell M =$ span of iterated Lie brackets of the X_j at x up to order ℓ
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    - $r \leq n \leq Q$, with strict inequalities in nonelliptic case
- example: homogeneous sub-Laplacians on stratified (Carnot) groups
    - $G \cong \mathfrak{g} = \bigoplus_{\ell=1}^{s} \mathfrak{g}_\ell$, $\quad \mathfrak{g}_{\ell+1} = [\mathfrak{g}_1, \mathfrak{g}_\ell]$, $\quad \{X_j\}_j$ basis of $\mathfrak{g}_1$, $\quad \mathscr{L} = -\sum_j X_j^2$
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- <u>bracket-generating condition</u>: the X_j and their iterated Lie brackets up to order s span the tangent space of M at each point
  - the minimum s is called the *step* ($s = 1$ in the elliptic case)
  - (Hörmander '67) $\mathscr{L}$ is hypoelliptic and satisfies subelliptic estimates
  - (Chow '39) points of M can be joined by horizontal curves
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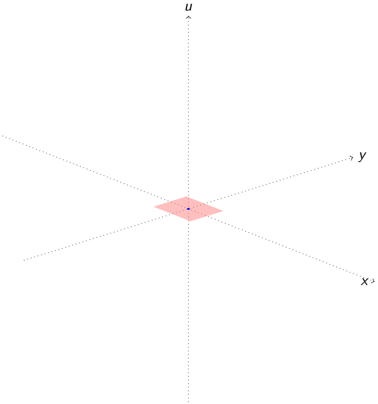

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Theorem 2 (M. & Müller & Nicolussi Golo, arXiv:1812.02671)

If, for some $p \in [1, \infty]$ and $\alpha \geq 0$, the estimate

$$\|(1 + t^2 \mathscr{L})^{-\alpha/2} \cos(t\sqrt{\mathscr{L}})\|_{p \to p} \lesssim 1$$

holds for all small t, then $\alpha \geq (n-1)|1/2 - 1/p|$.

- $\chi \in C_c^\infty((0, \infty))$ nontrivial cutoff

- M n-manifold with smooth measure μ
- $\mathscr{L} = \mathcal{H}(x, D) +$ first order terms
  - nonnegative selfadjoint second-order differential operator on M
  - satisfying Hörmander's condition at a point $x_0 \in M$

Theorem 1 (M. & Müller & Nicolussi Golo, arXiv:1812.02671)

If, for some $p \in [1, \infty]$ and $s \geq 0$, the estimate

$$\|F(\mathscr{L})\|_{p \to p} \lesssim \sup_{t \geq 0} \|F(t\cdot)\,\chi\|_{L^\infty_s}$$

holds for all $F \in \mathcal{S}(\mathbb{R})$, then $s \geq n|1/2 - 1/p|$. In particular, $\varsigma(\mathscr{L}) \geq n/2$.

Theorem 2 (M. & Müller & Nicolussi Golo, arXiv:1812.02671)

If, for some $p \in [1, \infty]$ and $\alpha \geq 0$, the estimate

$$\|\chi(t\sqrt{\mathscr{L}}/\lambda)\cos(t\sqrt{\mathscr{L}})\|_{p \to p} \lesssim \lambda^\alpha$$

holds for all small t and large λ, then $\alpha \geq (n-1)|1/2 - 1/p|$.

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FIO representation of the wave propagator
for elliptic $\mathscr{L} = \mathcal{H}(x, D) +$ first order terms

- locally and for small times,

$$\cos(t\sqrt{\mathscr{L}}) = Q(t) + Q(-t) + \text{smoothing},$$

where

$$Q(t)f(x) = \int e^{i\phi(t,x,y,\xi)}\, q(t,x,y,\xi)\, f(y)\, dy\, d\xi$$

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- the general case follows by "nonisotropic transplantation" (M. '17)