# Wiener-type characterization of boundary regularity via Gaussian bounds

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- Description of the class of operators
- Wiener and Wiener-Landis criteria: literature and main result
- Sketch of the proof and applications

joint work with F. Uguzzoni (Bologna),
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## Looking for Wiener criteria

The aim of this research is to establish Wiener-type criteria for evolution operators such as

$$\sum_{j=1}^{m} X_j^2 - \partial_t$$

where $X_j$ are smooth vector fields in $\mathbb{R}^N$ satisfying the Hörmander condition.

More generally, consider the following linear second order operator

$$\mathcal{H} = \sum_{i,j=1}^{N} q_{i,j}(z)\partial^2_{x_i,x_j} + \sum_{k=1}^{N} q_k(z)\partial_{x_k} - \partial_t,$$

defined in the strip of $\mathbb{R}^{N+1}$

$$S = \{z = (x,t) : x \in \mathbb{R}^N,\ T_1 < t < T_2\}, -\infty \le T_1 < T_2 \le +\infty.$$

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Assumptions:

- the coefficients $q_{i,j} = q_{j,i}, q_k$ are of class $C^\infty$;
- the quadratic form $q_{\mathcal{H}}(z,\xi) = \sum_{i,j=1}^N q_{i,j}(z)\xi_i\xi_j$ is
  nonnegative definite, i.e. $q_{\mathcal{H}}(z,\cdot) \geq 0$ for every $z \in S$,
  and not totally degenerate, i.e. $q_{\mathcal{H}}(z,\cdot) \not\equiv 0$ for every $z \in S$;
- $\mathcal{H}$ and its adjoint $\mathcal{H}^*$ are $C^\infty$-hypoelliptic;
- there exists a global *fundamental solution* $(z,\zeta) \mapsto \Gamma(z,\zeta)$ smooth out of the diagonal of $S \times S$
  (in the sense $\Gamma \in L^1_{\mathrm{loc}}$, $\mathcal{H}(\Gamma(\cdot,\zeta)) = -\delta_\zeta$ for any $\zeta$, and

$$\int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\,\varphi(\xi)\,d\xi \to \varphi(x_0)$$

  as $x \to x_0$ and $t \searrow \tau \in ]T_1, T_2[$ (or $\tau \nearrow t \in ]T_1, T_2[$), for every $\varphi \in C_0(\mathbb{R}^N)$ and for every $x_0 \in \mathbb{R}^N$).

MAIN ASSUMPTION:

Given a metric $d : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, we call $d$-Gaussian (of exponent $a > 0$) any function

$$\mathbb{G}_a(z,\zeta) = \mathbb{G}_a(x,t,\xi,\tau) = \begin{cases} 0 & \text{if } t \leq \tau, \\ \frac{1}{|B_d(x,\sqrt{t-\tau})|} \exp\left(-a\frac{d^2(x,\xi)}{t-\tau}\right) & \text{if } t > \tau. \end{cases}$$

We assume the existence of a distance $d$ in $\mathbb{R}^N$ such that the following Gaussian estimates for $\Gamma$ hold

$$\text{(H)} \qquad \frac{1}{\Lambda}\mathbb{G}_{b_0}(z,\zeta) \leq \Gamma(z,\zeta) \leq \Lambda\mathbb{G}_{a_0}(z,\zeta), \quad \forall z,\zeta \in S,$$

for suitable positive constants $a_0$, $b_0$, and $\Lambda$.

We shall make the following assumptions on the metric space $(\mathbb{R}^N, d)$:

(D1) The $d$-topology is the Euclidean topology. Moreover $(\mathbb{R}^N, d)$ is complete and, for every fixed $x \in \mathbb{R}^N$, $d(x,\xi) \to \infty$ if and only if $|\xi| \to \infty$.

(D2) $(\mathbb{R}^N, d)$ is doubling w.r.t. the Lebesgue measure, i.e. $\exists c_d > 1$ such that

$$|B(x,2r)| \leq c_d |B(x,r)|, \quad \forall x \in \mathbb{R}^N, \ \forall r > 0.$$

(D3) $(\mathbb{R}^N, d)$ has the *segment property*, i.e., for every $x, y \in \mathbb{R}^N$ there exists a continuous path $\gamma : [0,1] \to \mathbb{R}^N$ such that $\gamma(0) = x$, $\gamma(1) = y$ and

$$d(x,y) = d(x,\gamma(t)) + d(\gamma(t),y) \quad \forall t \in [0,1].$$

The Hörmander case:

Our results apply in particular to degenerate parabolic operators of Hörmander type

$$\sum_{i,j=1}^{m} a_{i,j}(x,t)X_iX_j + \sum_{k=1}^{m} a_k(x,t)X_k - \partial_t$$

where $\mathcal{X} = \{X_1, \ldots, X_m\}$ is a system of smooth vector fields satisfying the Hörmander rank condition in $\mathbb{R}^N$, and $A(z) = (a_{i,j})$ is uniformly positive definite.

To see this, one should keep in mind that we want to study boundary value problems in bounded open sets and we are not effected by large-scale geometries.

Under these assumptions, it is possible to develop a Potential Analysis for $\mathcal{H}$ [Lanconelli-Uguzzoni 2010]:

the operator $\mathcal{H}$ endows the strip $S$ with a structure of $\beta$-harmonic space satisfying the Doob convergence property.

As a consequence, for any bounded open set $\Omega$ with $\overline{\Omega} \subseteq S$, the Dirichlet problem

$$\begin{cases} \mathcal{H}u = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = \varphi \end{cases}$$

has a generalized solution $H^{\Omega}_{\varphi}$, in the Perron-Wiener sense, for every continuous function $\varphi : \partial\Omega \to \mathbb{R}$.

**Definition**

*Let $\Omega$ be a bounded open set with $\overline{\Omega} \subseteq S$. A point $z_0 \in \partial\Omega$ is called $\mathcal{H}$-regular if*

$$\lim_{\Omega \ni z \to z_0} H_{\varphi}^{\Omega}(z) = \varphi(z_0) \quad \textit{for every } \varphi \in C(\partial\Omega).$$

Wiener criteria are tests to prove or disprove regularity by checking whether a suitable series (which classically involves capacitary terms) is divergent or convergent.

Classical Wiener criterion: Wiener (1924)

Consider $\Delta$ in $\mathbb{R}^n \supset\supset \Omega$, $x_0 \in \partial\Omega$. For $\mu \in ]0,1[$, define

$$\Omega_k^\Delta(x_0) = \big(B(x_0,\mu^k) \setminus B(x_0,\mu^{k+1})\big) \setminus \Omega.$$

Then

$$x_0 \text{ is } \Delta-\text{regular} \quad \Leftrightarrow \quad \sum_{k=1}^{\infty} \frac{\mathrm{cap}(\Omega_k^\Delta(x_0))}{\mu^{k(n-2)}} = +\infty$$

Littman-Stampacchia-Weinberger (1963):

$$L = \mathrm{div}(A(x)\nabla), \qquad A \text{ uniformly elliptic, in } \mathbb{R}^n \supset\supset \Omega, \quad x_0 \in \partial\Omega.$$

$$x_0 \text{ is } L-\text{regular} \quad \Leftrightarrow \quad \sum_{k=1}^{\infty} \frac{\mathrm{cap}(\Omega_k^{\Delta}(x_0))}{\mu^{k(n-2)}} = +\infty$$

$$\Leftrightarrow \quad x_0 \text{ is } \Delta - \text{regular}$$

There is a long literature around Wiener criteria for elliptic/degenerate-elliptic operators.

Heat equation: Lanconelli (1973) for $\Rightarrow$, Evans-Gariepy (1982) for $\Leftarrow$

Consider $\Delta - \partial_t$ in $\mathbb{R}^{n+1} \supset\supset \Omega$, $z_0 = (x_0, t_0) \in \partial\Omega$. For $\lambda \in (0, 1)$, define

$$\Omega_k(z_0) = \left\{ \zeta \notin \Omega \,:\, \frac{1}{\lambda^k} \leq G(z_0, \zeta) \leq \frac{1}{\lambda^{k+1}} \right\}.$$

Then,

$$z_0 \text{ is } (\Delta - \partial_t) - \text{regular} \qquad \Leftrightarrow \qquad \sum_{k=1}^{\infty} \frac{\mathrm{cap}(\Omega_k(z_0))}{\lambda^k} = +\infty$$

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For any given a compact set $F \subset \mathbb{R}^{n+1}$, one can define

$$\mathrm{cap}(F) = \sup \Big\{ \mu(F) \,:\, \mu \in \mathcal{M}^+(F), \text{ and } \Gamma * \mu(z) := \int \Gamma(z,\zeta) d\mu(\zeta) \le 1 \quad \forall z \in \mathbb{R}^{N+1} \Big\}$$

Petrowsky 1935:

$$\alpha \neq \beta, \quad 0 < \alpha, \beta$$

$$\text{regularity for } \alpha\,\Delta - \partial_t \qquad \not\equiv \qquad \text{regularity for } \beta\,\Delta - \partial_t$$

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Variable coefficients case: Garofalo-Lanconelli (1988)

Consider

$$\mathrm{div}(A(x,t)\nabla) - \partial_t,$$

$A$ uniformly positive definite with smooth entries, in $\mathbb{R}^{n+1} \supset\supset \Omega$, $z_0 = (x_0, t_0) \in \partial\Omega$. For $\lambda \in (0,1)$, denote

$$\Omega_k^A(z_0) = \left\{ \zeta \notin \Omega \,:\, \frac{1}{\lambda^k} \leq \Gamma^A(z_0,\zeta) \leq \frac{1}{\lambda^{k+1}} \right\}.$$

Then,

$$z_0 \text{ is } (\mathrm{div}(A(x,t)\nabla) - \partial_t)\text{-regular} \quad \Leftrightarrow \quad \sum_{k=1}^{\infty} \frac{\mathrm{cap}(\Omega_k^A(z_0))}{\lambda^k} = +\infty.$$

Degenerate-parabolic: Garofalo-Segala (1990) for the heat operator in the Heisenberg group (Rotz (2016) for H-type groups)

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The Wiener criterion can also be read as follows:

consider the balayage $V_{\Omega_k(z_0)}$ of the compact sets $\Omega_k(z_0)$ and their Riesz-measures $\mu_{\Omega_k(z_0)}$ (for which $\mathcal{H} V_{\Omega_k(z_0)} = -\mu_{\Omega_k(z_0)}$), one has the representation

$$V_{\Omega_k(z_0)} = \Gamma * \mu_{\Omega_k(z_0)}.$$

Since $\mu_{\Omega_k(z_0)}(\Omega_k(z_0)) \sim \mathrm{cap}(\Omega_k(z_0))$, one can see that

$$\sum_{k=1}^{\infty} \frac{\mathrm{cap}(\Omega_k(z_0))}{\lambda^k} \sim \sum_{k=1}^{\infty} \Gamma * \mu_{\Omega_k(z_0)}(z_0).$$

Landis criterion for the Heat equation: Landis (1969)

Consider $\Delta - \partial_t$ in $\mathbb{R}^{n+1} \supset\supset \Omega$, $z_0 = (x_0, t_0) \in \partial\Omega$. There exists a sequence $\{\alpha(k)\}_{k\in\mathbb{N}}$ (growing fast at $\infty$) such that, if we define

$$\Omega_k^c(z_0) = \left\{ \zeta \notin \Omega \,:\, \left(\frac{1}{\lambda}\right)^{\alpha(k)} \leq G(z_0, \zeta) \leq \left(\frac{1}{\lambda}\right)^{\alpha(k+1)} \right\},$$

we have

$$z_0 \text{ is } (\Delta - \partial_t) - \text{regular} \quad \Leftrightarrow \quad \sum_{k=1}^{\infty} \Gamma * \mu_{\Omega_k^c(z_0)}(z_0) = +\infty.$$

The Wiener criterion proved by Landis provides in fact a true characterization for the regularity of boundary points. On the other hand, if one tries to read it with respect to capacitary terms, one recognizes that

$$\sum_{k=1}^{\infty} \frac{\mathrm{cap}(\Omega_k^c(z_0))}{\lambda^{\alpha(k)}} \lesssim \sum_{k=1}^{\infty} \Gamma * \mu_{\Omega_k^c(z_0)}(z_0) \lesssim \sum_{k=1}^{\infty} \frac{\mathrm{cap}(\Omega_k^c(z_0))}{\lambda^{\alpha(k+1)}}.$$

This produces a mismatch between the necessary and the sufficient condition (unless $\alpha(k)$ is linear).

We proved a Wiener-type characterization for the $\mathcal{H}$-regularity of boundary points in the spirit of the results by Landis. We also established an explicit behavior for the sequence $\alpha(k)$.

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Let us fix

$$\Omega_k^c(z_0) = \left\{ \zeta \notin \Omega \,:\, \left(\frac{1}{\lambda}\right)^{k \log(k)} \le \Gamma(z_0, \zeta) \le \left(\frac{1}{\lambda}\right)^{(k+1)\log(k+1)} \right\}.$$

We have the following

**Theorem [T. - Uguzzoni (2020)]**

*Let $\Omega$ be a bounded open set with $\overline{\Omega} \subseteq S$, and let $z_0 \in \partial\Omega$. Then*

$$z_0 \;\; \textit{is} \;\; \mathcal{H}-\textit{regular} \qquad \Leftrightarrow \qquad \sum_{k=1}^{\infty} \Gamma * \mu_{\Omega_k^c(z_0)}(z_0) = +\infty.$$

## Remarks on the proof

We followed a different strategy with respect to Landis.
Landis' proof relied in fact on a control of the oscillation at the boundary by making use of explicit barrier function.

We adopted the same strategy as in [Kogoj-Lanconelli-T., (2018)], where we proved a Wiener-Landis test for Kolmogorov operator. In that situation, the kernel $\Gamma$ is known explicit, and it appeared there the choice $\alpha(k) = k \log(k)$.

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## Applications

Let us apply the sufficient criterion in concrete situation. Consider

$$\mathcal{H}_0 = \Delta_{\mathbb{G}} - \partial_t,$$

where $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_r\right)$ is a Carnot group. Let $Q$ be the homogeneous dimension, and $d$ an homogeneous distance.

### Corollary

*Consider a bounded open set $\Omega$ in $\mathbb{R}^{N+1}$, and $z_0 \in \partial\Omega$. There exists a positive constant $C^* = C^*(b_0, Q)$ such that, if we have*

$$\Big\{(x,t) \in \mathbb{R}^{N+1} \,:\, d^2(x,x_0) \geq C(t_0 - t)\log\log\left(\frac{1}{t_0 - t}\right),$$
$$\text{for } t \in \left(t_0 - \min\{r_0^2, e^{-1}\}, t_0\right)\Big\} \subset \mathbb{R}^{N+1} \smallsetminus \Omega$$

*for some $r_0 > 0$ and $0 < C < C^*$, then the point $z_0$ is $\mathcal{H}_0$-regular for $\partial\Omega$.*

## Remarks

The stronger condition

$$\left\{(x,t) \in \mathbb{R}^{N+1} \,:\, d^2(x,x_0) \geq C(t_0 - t)\right\} \subset \mathbb{R}^{N+1} \setminus \Omega$$

is a parabolic-cone condition: under this condition, the regularity of $z_0 = (x_0, t_0)$ was known [Lanconelli-Uguzzoni (2010)], and also a $C^\alpha$-modulus of continuity for the solution ([Lanconelli-T.-Uguzzoni (2017)]).

The log log-paraboloid condition is sharp in the following sense: for the classical heat equation is known that the point is NOT regular for a boundary $\partial\Omega$ with that log log-profile if $C$ is big enough. This is precisely the nature of Petrowski's counterexamples.
We showed that

$$\sum_k^\infty \Gamma * \mu_{\Omega_k^c(z_0)}(z_0) \gtrsim \sum_k^\infty \frac{1}{\alpha(k)} = \sum_k^\infty \frac{1}{k \log k}.$$

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# Thanks for the attention