The critical exponent for the damped wave equation in the Heisenberg group

Alessandro Palmieri

University of Pisa

Dissipative and subelliptic PDEs
Centro di Ricerca Matematica "E. De Giorgi", Scuola Normale Superiore
February 10, 2020

Joint work with Vladimir Georgiev (University of Pisa)

Introduction

It is known by the classical works of Matsumura (1976), Todorova-Yordanov (2001) and Zhang (2001), that the critical exponent of the Cauchy problem for the semilinear damped wave equation in the Euclidean setting

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^p, & x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases}$$

is the same one as for the semilinear heat equation, namely, the Fujita exponent $p_{\mathrm{Fuj}}(n) \doteq 1 + 2/n$. This is due to the presence of diffusion phenomena among the corresponding linear equations.
In this talk we are going to focus on the analogous semilinear Cauchy problem, but in the Heisenberg group, providing the corresponding critical exponent.

Short recap on Heisenberg group

The Heisenberg group is the Lie group $\mathbb{R}^{2n+1} = \mathbf{H}_n$ endowed with the multiplication law

$$(x, y, \tau) \circ (x', y', \tau') = (x + x', y + y', \tau + \tau' + \tfrac{1}{2}(x \cdot y' - y \cdot y').$$

A system of left-invariant vector fields that span the lie algebra $\mathfrak{h}_n = \mathrm{Lie}(\mathbf{H}_n)$ is given by

$$X_j \doteq \partial_{x_j} - \frac{y_j}{2}\partial_\tau,\ Y_j \doteq \partial_{y_j} + \frac{x_j}{2}\partial_\tau,\ \partial_\tau \quad (1 \leqslant j \leqslant n)$$

so the Lie algebra admits the stratification $\mathfrak{h}_n = V_1 \oplus V_2$, where $V_1 \doteq \mathrm{span}\{X_j, Y_j\}_{1 \leqslant j \leqslant n}$ is the so-called *horizontal layer* and $V_2 \doteq \mathrm{span}\{\partial_\tau\}$. The vector fields in V_1 are called *horizontal vector fields*.

Therefore, $\mathbf{H}_n$ is a *stratified Lie group* of step 2 with *homogeneous dimension* $\mathbb{Q} = 2n + 2$.

Let $u \in \mathscr{C}^1(\mathbf{H}_n)$, then the *horizontal gradient* of u is

$$\nabla_{\mathrm{H}} u \doteq \sum_{j=1}^{n} (X_j u) X_j + (Y_j u) Y_j.$$

Let $u \in \mathscr{C}^2(\mathbf{H}_n)$, then the *sub-Laplacian* of u is

$$\Delta_{\mathrm{H}} u \doteq \sum_{j=1}^{n} X_j^2 u + Y_j^2 u.$$

As in the Euclidean framework, we might introduce $X_j u$ and $Y_j u$ in the distributional sense by using $\mathscr{C}_0^\infty(\mathbf{H}_n)$ functions. So, we shall consider as Sobolev space

$$H^1(\mathbf{H}_n) \doteq \{u \in L^2(\mathbf{H}_n) : X_j u, Y_j u \in L^2(\mathbf{H}_n) \text{ for any } j = 1, \ldots, n\}$$

Our problem

We are going to investigate the Cauchy problem for the semilinear damped wave equation in the Heisenberg group $\mathbf{H}_n$, namely

$$\begin{cases} u_{tt} - \Delta_{\mathrm{H}} u + u_t = |u|^p, & \eta \in \mathbf{H}_n,\ t > 0, \\ u(0,x) = u_0(x), & \eta \in \mathbf{H}_n, \\ u_t(0,x) = u_1(x), & \eta \in \mathbf{H}_n. \end{cases} \tag{CP}$$

We shall prove that the critical exponent for (CP) is the Fujita-type exponent $p_{\mathrm{Fuj}}(\mathbb{Q}) = 1 + 2/\mathbb{Q}$.

We point out that recently it has been proven even for the semilinear heat equation in the Heisenberg group that $p_{\mathrm{Fuj}}(\mathbb{Q})$ is the critical exponent by Ruzhansky-Yessirkegenov (in the more general frame of unimodular Lie group with polynomial volume growth) and by Georgiev-P (in the Heisenberg group with lifespan estimates in the subcritical and critical cases).

In order to show that $p_{\mathrm{Fuj}}(\mathbb{Q})$ is the critical exponent for (CP), we have to show a blow-up result for $1 < p \leqslant p_{\mathrm{Fuj}}(\mathbb{Q})$ and global existence result for small data solutions for $p > p_{\mathrm{Fuj}}(\mathbb{Q})$.

The following blow-up result can be proved by using the so-called *test function method* developed by Mitidieri-Pohozaev (and applied to the semilinear damped wave model by Zhang). This method is based on a scaling argument, so for the scaling in Heisenberg group we need to employ the anisotropic dilations.
On the other hand, for the global existence result we will work with exponentially weighted Sobolev spaces. In particular, the decay estimates for the corresponding homogeneous problem are derived by using phase space analysis in $\mathbf{H}_n$.

Blow-up result

Theorem

Let $n \geq 1$. Let $u_0, u_1 \in L^1(\mathbf{H}_n)$ such that

$$\liminf_{R\to\infty} \int_{\mathscr{D}_R} \big(u_0(\eta) + u_1(\eta)\big)\mathrm{d}\eta > 0,$$

where $\mathscr{D}_R \doteq B_n(R) \times B_n(R) \times [-R^2, R^2]$.
Let us assume that $u \in L^p_{\mathrm{loc}}([0,T) \times \mathbb{R}^n)$ is a weak solution to (CP), *with lifespan $T > 0$. If $1 < p \leq p_{\mathrm{Fuj}}(\mathbb{Q})$, then $T < \infty$, that is, the solutions u blows up in finite time.*

Global existence result

Before stating the main result concerning the global existence of small data solutions, we introduce the main tools that are used in the proof.

The first fundamental result for the proof (definition of the space for the solution) is given by the following decay estimate for the homogeneous Cauchy problem

$$\begin{cases} u_{tt} - \Delta_{\mathrm{H}} u + u_t = 0, & \eta \in \mathbf{H}_n,\ t > 0, \\ u(0,x) = u_0(x), & \eta \in \mathbf{H}_n, \\ u_t(0,x) = u_1(x), & \eta \in \mathbf{H}_n. \end{cases} \tag{hCP}$$

Theorem (Decay estimates)

Let us assume $(u_0, u_1) \in (H^1(\mathbf{H}_n) \cap L^1(\mathbf{H}_n)) \times (L^2(\mathbf{H}_n) \cap L^1(\mathbf{H}_n))$*.*
Let $u \in \mathscr{C}([0,\infty), H^1(\mathbf{H}_n)) \cap \mathscr{C}^1([0,\infty), L^2(\mathbf{H}_n))$ *solve the Cauchy problem* (hCP)*. Then, the following decay estimates are satisfied*

$$\|u(t,\cdot)\|_{L^2(\mathbf{H}_n)} \leq C(1+t)^{-\frac{\mathbb{Q}}{4}} \|(u_0,u_1)\|_{L^1(\mathbf{H}_n)\cap L^2(\mathbf{H}_n)}$$

$$\|\nabla_{\mathrm{H}} u(t,\cdot)\|_{L^2(\mathbf{H}_n)} \leq C(1+t)^{-\frac{\mathbb{Q}}{4}-\frac{1}{2}} \|(u_0,u_1)\|_{(H^1(\mathbf{H}_n)\cap L^1(\mathbf{H}_n))\times(L^2(\mathbf{H}_n)\cap L^1(\mathbf{H}_n))}$$

$$\|\partial_t u(t,\cdot)\|_{L^2(\mathbf{H}_n)} \leq C(1+t)^{-\frac{\mathbb{Q}}{4}-1} \|(u_0,u_1)\|_{(H^1(\mathbf{H}_n)\cap L^1(\mathbf{H}_n))\times(L^2(\mathbf{H}_n)\cap L^1(\mathbf{H}_n))}$$

for any $t \geq 0$*. Furthermore, if we assume just* $(u_0,u_1) \in H^1(\mathbf{H}_n) \times L^2(\mathbf{H}_n)$*, that is, we do not require additional* $L^1(\mathbf{H}_n)$ *regularity for the Cauchy data, then the following estimates are satisfied*

$$\|u(t,\cdot)\|_{L^2(\mathbf{H}_n)} \leq C\|(u_0,u_1)\|_{L^2(\mathbf{H}_n)}$$

$$\|\nabla_{\mathrm{H}} u(t,\cdot)\|_{L^2(\mathbf{H}_n)} \leq C(1+t)^{-\frac{1}{2}} \|(u_0,u_1)\|_{H^1(\mathbf{H}_n)\times L^2(\mathbf{H}_n)}$$

$$\|\partial_t u(t,\cdot)\|_{L^2(\mathbf{H}_n)} \leq C(1+t)^{-1} \|(u_0,u_1)\|_{H^1(\mathbf{H}_n)\times L^2(\mathbf{H}_n)}$$

for any $t \geq 0$*. Here* $C > 0$ *is a universal constant.*

We underline that the previous decay estimates are completely analogous to the ones derived in the Euclidean framework in the pioneering paper of Matsumura (1976).

Another important tool that allows us to handle the nonlinear term is a Gagliardo-Nirenberg type inequality in $\mathbf{H}_n$:

Lemma (Gagliardo-Nirenberg inequality)

Let $n \geq 1$. Let us consider $2 \leq q \leq 2 + \frac{2}{n} = \frac{2\mathbb{Q}}{\mathbb{Q}-2}$. Then, the following Gagliardo-Nirenberg inequality holds

$$\|v\|_{L^q(\mathbf{H}_n)} \leq C \, \|\nabla_{\mathrm{H}} v\|^{\theta(q)}_{L^2(\mathbf{H}_n)} \|v\|^{1-\theta(q)}_{L^2(\mathbf{H}_n)}$$

for any $v \in H^1(\mathbf{H}_n)$, where C is a nonnegative constant and $\theta(q) \in [0, 1]$ is defined by

$$\theta(q) \doteq \mathbb{Q} \left(\frac{1}{2} - \frac{1}{q} \right).$$

If we worked only with the classical energy spaces (i.e. without considering exponentially weighted Sobolev spaces), we would apply GN inequality to estimate both the $L^p(\mathbf{H}_n)$-norm of u and $L^{2p}(\mathbf{H}_n)$-norm of u. But then we would have a not empty range for p only for $n = 1$ (furthermore, in this case the range for p is reduce to $\{2\}$).

Therefore, we introduce the Sobolev spaces L^2 and H^1 with exponential weight $\mathrm{e}^{\psi(t,\cdot)}$

$$L^2_{\psi(t,\cdot)}(\mathbf{H}_n) \doteq \{v \in L^2(\mathbf{H}_n) : \|\mathrm{e}^{\psi(t,\cdot)} v\|_{L^2(\mathbf{H}_n)} < \infty\},$$
$$H^1_{\psi(t,\cdot)}(\mathbf{H}_n) \doteq \{v \in H^1(\mathbf{H}_n) : \|\mathrm{e}^{\psi(t,\cdot)} v\|_{L^2(\mathbf{H}_n)} + \|\mathrm{e}^{\psi(t,\cdot)} \nabla_{\mathrm{H}} v\|_{L^2(\mathbf{H}_n)} < \infty\},$$

where

$$\psi(t,\eta) \doteq \frac{|x|^2 + |y|^2 + 4|\tau|}{8(1+t)} \quad \text{for any } \eta = (x, y, \tau) \in \mathbf{H}_n.$$

In particular, as space for the Cauchy data we consider

$$\mathscr{A}(\mathbf{H}_n) \doteq H^1_{\psi(0,\cdot)}(\mathbf{H}_n) \times L^2_{\psi(0,\cdot)}(\mathbf{H}_n).$$

Fundamental properties of ψ

$$|\nabla_{\mathrm{H}}\psi(t,\eta)|^2 + \psi_t(t,\eta) \leq 0$$

$$\Delta_{\mathrm{H}}\psi(t,\eta) = \frac{n}{2(1+t)} + \frac{|x|^2+|y|^2}{4(1+t)}\delta_0(\tau)$$

for any $t \geq 0$ and any $\eta = (x, y, \tau) \in \mathbf{H}_n$, where $\delta_0(\tau)$ denotes the Dirac delta in 0 with respect to the τ variable.

These properties of the function $\psi(t,\cdot)$ are essential to prove the next a-priori estimate.

Lemma

Let $n \geq 1$ and $p > 1$ such that $p \leqslant p_{\mathrm{GN}}(\mathbb{Q}) \doteq \frac{\mathbb{Q}}{\mathbb{Q}-2}$.
Let $(u_0, u_1) \in \mathscr{A}(\mathbf{H}_n)$. If u solves (hCP), then, the following energy estimate holds for any $t \in [0, T)$ and for an arbitrary small $\delta > 0$

$$\mathscr{E}_\psi[u](t) \lesssim \mathscr{E}_\psi[u](0) + \mathscr{E}_\psi[u](0)^{\frac{p+1}{2}} + \left(\sup_{s\in[0,t]} (1+s)^\delta \|\mathrm{e}^{\left(\frac{2}{p+1}+\delta\right)\psi(s,\cdot)} u(s,\cdot)\|_{L^{p+1}(\mathbf{H}_n)} \right)^{p+1},$$

where

$$\mathscr{E}_\psi[u](t) \doteq \int_{\mathbf{H}_n} \mathrm{e}^{2\psi(t,\eta)} \left(|u_t(t,\eta)|^2 + |\nabla_{\mathrm{H}} u(t,\eta)|^2 \right) \mathrm{d}\eta.$$

Now we can state the global existence result.

Theorem

Let $n \geq 1$. Let us consider $1 < p \leqslant p_{\mathrm{GN}}(\mathbb{Q})$ such that $p > p_{\mathrm{Fuj}}(\mathbb{Q})$. Then, there exists $\varepsilon_0 > 0$ such that for any initial data

$$(u_0, u_1) \in \mathscr{A}(\mathbf{H}_n) \quad \textit{satisfying} \quad \|(u_0, u_1)\|_{\mathscr{A}(\mathbf{H}_n)} \leqslant \varepsilon_0$$

there is a unique solution $u \in \mathscr{C}([0,\infty), H^1_{\psi(t,\cdot)}(\mathbf{H}_n)) \cap \mathscr{C}^1([0,\infty), L^2_{\psi(t,\cdot)}(\mathbf{H}_n))$ to the Cauchy problem (CP). Moreover, u satisfies the following estimates

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbf{H}_n)} &\lesssim (1+t)^{-\frac{\mathbb{Q}}{4}} \|(u_0,u_1)\|_{\mathscr{A}(\mathbf{H}_n)}, \\
\|\nabla_{\mathrm{H}} u(t,\cdot)\|_{L^2(\mathbf{H}_n)} &\lesssim (1+t)^{-\frac{\mathbb{Q}}{4}-\frac{1}{2}} \|(u_0,u_1)\|_{\mathscr{A}(\mathbf{H}_n)}, \\
\|u_t(t,\cdot)\|_{L^2(\mathbf{H}_n)} &\lesssim (1+t)^{-\frac{\mathbb{Q}}{4}-1} \|(u_0,u_1)\|_{\mathscr{A}(\mathbf{H}_n)}, \\
\|\mathrm{e}^{\psi(t,\cdot)} \nabla_{\mathrm{H}} u(t,\cdot)\|_{L^2(\mathbf{H}_n)} &\lesssim \|(u_0,u_1)\|_{\mathscr{A}(\mathbf{H}_n)}, \\
\|\mathrm{e}^{\psi(t,\cdot)} u_t(t,\cdot)\|_{L^2(\mathbf{H}_n)} &\lesssim \|(u_0,u_1)\|_{\mathscr{A}(\mathbf{H}_n)}
\end{aligned}$$

for any $t \geq 0$.

Decay Estimates for (hCP)

In this final part, we want to sketch the main ideas to prove the decay estimates for the homogeneous problem (hCP). We follow the approach introduced by Ruzhansky-Tokmagambetov (2018).

As we have already mentioned, the main tool to prove these estimates is the Fourier group transform on $\mathbf{H}_n$. We employ the following realization of Schrödinger representations $\{\pi_\lambda\}_{\lambda\in\mathbb{R}^*}$

$$\pi_\lambda : \mathbf{H}_n \to \mathcal{U}(L^2(\mathbb{R}^n))$$
$$\pi_\lambda(x,y,\tau)\phi(w) = \mathrm{e}^{i\lambda\left(\tau+\frac{1}{2}x\cdot y\right)}\mathrm{e}^{i\,\mathrm{sign}(\lambda)\sqrt{|\lambda|}\,y\cdot w}\phi\big(w+\sqrt{|\lambda|}x\big)$$

for any $\lambda \in \mathbb{R}^* \doteq \mathbb{R}\setminus\{0\}$, $(x,y,\tau)\in\mathbf{H}_n$, $\phi\in L^2(\mathbb{R}^n)$ and $w\in\mathbb{R}^n$, where $\mathcal{U}(L^2(\mathbb{R}^n)) \subset \mathscr{L}(L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n))$ is the set of unitary bounded operators on $L^2(\mathbb{R}^n)$ (cf. the monograph by Fisher-Ruzhansky, 2016).

In the Heisenberg group we have an explicit description of the unitary dual group (strongly continuous unitary representations up to intertwining operators) $\widehat{\mathbf{H}}_n \simeq \mathbb{R}^*$ and of the Plancherel measure

$$\mathrm{d}\mu(\pi_\lambda) = c_n |\lambda|^n \mathrm{d}\lambda,$$

where $c_n \doteq (2\pi)^{-(3n+1)}$ in this setting.

If $v \in L^1(\mathbf{H}_n)$, the group Fourier transform of $\widehat{v}$ is the family of bounded operators on $L^2(\mathbb{R}^n)$

$$\widehat{v}(\lambda) \doteq \int_{\mathbf{H}_n} v(\eta)\pi_\lambda^*(\eta)\,\mathrm{d}\eta \in \mathscr{L}(L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)),\ \lambda \in \mathbb{R}^*.$$

Since Schrödinger representation are unitary, we see immediately that

$$\sup_{\lambda\in\mathbb{R}^*} \|\widehat{v}(\lambda)\|_{\mathscr{L}(L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n))} \leqslant \|v\|_{L^1(\mathbf{H}_n)}.$$

If $v \in L^2(\mathbf{H}_n)$, then Plancherel formula holds

$$\begin{aligned}
\|v\|^2_{L^2(\mathbf{H}_n)} &= c_n \int_{\mathbb{R}^*} \|\widehat{v}(\lambda)\|^2_{\mathrm{HS}[L^2(\mathbb{R}^n)]} |\lambda|^n \, \mathrm{d}\lambda \\
&= c_n \int_{\mathbb{R}^*} \sum_{k \in \mathbb{N}^n} \|\widehat{v}(\lambda) e_k\|^2_{L^2(\mathbb{R}^n)} |\lambda|^n \, \mathrm{d}\lambda \\
&= c_n \int_{\mathbb{R}^*} \sum_{k,\ell \in \mathbb{N}^n} \left| \left(\widehat{v}(\lambda) e_k, e_\ell \right)_{L^2(\mathbb{R}^n)} \right|^2 |\lambda|^n \, \mathrm{d}\lambda,
\end{aligned}$$

where $\{e_k\}_{k \in \mathbb{N}^n}$ is the orthonormal basis of $L^2(\mathbb{R}^n)$ given by Hermite functions.

We chose $\{e_k\}_{k \in \mathbb{N}^n}$ as basis of $L^2(\mathbb{R}^n)$ since its elements form a complete system of eigenvalues for the harmonic oscillator $\mathrm{H}_w \doteq -\Delta + |w|^2$ on $\mathbb{R}^n$.

Why is it important to "diagonalize" the harmonic oscillator? Because the action of the infinitesimal representation of π_λ on the sub-Laplacian is

$$\mathrm{d}\pi_\lambda(\Delta_{\mathsf{H}}) = -|\lambda| \mathrm{H}_w.$$

This follows from the action of the infinitesimal representation of π_λ on the generators of the horizontal layer of $\mathfrak{h}_n$

$$\mathrm{d}\pi_\lambda(X_j) = \sqrt{|\lambda|}\partial_{w_j},$$
$$\mathrm{d}\pi_\lambda(Y_j) = i \operatorname{sign}(\lambda)\sqrt{|\lambda|} w_j$$

for $j = 1, \ldots, n$.

After recalling all tools which are necessary to deal with the group Fourier transform in our framework, we may finally sketch the strategy to prove the decay estimates.

Let u be a solution of the homogeneous problem (hCP). Applying the group Fourier transform (with respect to the spatial variable η) to (hCP) we get an ODE with respect to t (depending on the parameter λ) in the Banach space $\mathscr{L}(L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n))$

$$\partial_t^2 \widehat{u}(t,\lambda) - \sigma_{\Delta_{\mathrm{H}}}(\lambda)\widehat{u}(t,\lambda) + \partial_t^2 \widehat{u}(t,\lambda) = 0.$$

Then, we project this equation by using the orthonormal system $\{e_k\}_{k\in\mathbb{N}^n}$. Since

$$\mathrm{H}e_k = \mu_k e_k, \qquad \mu_k \doteq 2|k| + n,\ \ k \in \mathbb{N}^n,$$

to determine the time-dependent function

$$\widehat{u}(t,\lambda)_{k,\ell} = \big(\widehat{u}(t,\lambda)e_k, e_\ell\big)_{L^2(\mathbb{R}^n)}$$

we have to solve the following Cauchy problem associated to the ODE depending on the parameters λ, k, ℓ

$$\begin{cases} \partial_t^2 \widehat{u}(t,\lambda)_{k,\ell} + \partial_t \widehat{u}(t,\lambda)_{k,\ell} + \mu_k|\lambda|\widehat{u}(t,\lambda)_{k,\ell} = 0, \quad t > 0, \\ \widehat{u}(0,\lambda)_{k,\ell} = \big(\widehat{u}_0(\lambda)e_k, e_\ell\big)_{L^2(\mathbb{R}^n)}, \\ \partial_t \widehat{u}(0,\lambda)_{k,\ell} = \big(\widehat{u}_1(\lambda)e_k, e_\ell\big)_{L^2(\mathbb{R}^n)}. \end{cases}$$

Solving explicitly the previous Cauchy problem, we find

$$\begin{aligned}\widehat{u}(t,\lambda)_{k,\ell} &= \left(\widehat{u}_0(\lambda)e_k, e_\ell\right)_{L^2(\mathbb{R}^n)} \mathrm{e}^{-\frac{t}{2}} F(t,\lambda,k) \\ &\qquad + \left((\widehat{u}_0(\lambda) + \tfrac{1}{2}\widehat{u}_1(\lambda))e_k, e_\ell\right)_{L^2(\mathbb{R}^n)} \mathrm{e}^{-\frac{t}{2}} G(t,\lambda,k),\end{aligned}$$

where

$$F(t,\lambda,k) \doteq \begin{cases} \cos\left(\sqrt{\mu_k|\lambda| - \frac{1}{4}}\, t\right) & \text{if } \ 4\mu_k|\lambda| > 1, \\ 1 & \text{if } \ 4\mu_k|\lambda| = 1, \\ \cosh\left(\sqrt{\frac{1}{4} - \mu_k|\lambda|}\, t\right) & \text{if } \ 4\mu_k|\lambda| < 1, \end{cases}$$

$$G(t,\lambda,k) \doteq \begin{cases} \dfrac{\sin\left(\sqrt{\mu_k|\lambda| - \frac{1}{4}}\, t\right)}{\sqrt{\mu_k|\lambda| - \frac{1}{4}}} & \text{if } \ 4\mu_k|\lambda| > 1, \\ t & \text{if } \ 4\mu_k|\lambda| = 1, \\ \dfrac{\sinh\left(\sqrt{\frac{1}{4} - \mu_k|\lambda|}\, t\right)}{\sqrt{\frac{1}{4} - \mu_k|\lambda|}} & \text{if } \ 4\mu_k|\lambda| < 1. \end{cases}$$

In order to estimate $\|u(t,\cdot)\|^2_{L^2(\mathbf{H}_n)}$ we apply Plancherel formula

$$\begin{aligned}
\|u(t,\cdot)\|^2_{L^2(\mathbf{H}_n)} &= c_n \sum_{k,\ell\in\mathbb{N}^n} \int_{\mathbb{R}^*} \left|\left(\widehat{u}(t,\lambda)e_k, e_\ell\right)_{L^2(\mathbb{R}^n)}\right|^2 |\lambda|^n \,\mathrm{d}\lambda \\
&= c_n \sum_{k,\ell\in\mathbb{N}^n} \int_{0<|\lambda|<\frac{1}{8\mu_k}} \left|\left(\widehat{u}(t,\lambda)e_k, e_\ell\right)_{L^2(\mathbb{R}^n)}\right|^2 |\lambda|^n \,\mathrm{d}\lambda \\
&\quad + c_n \sum_{k,\ell\in\mathbb{N}^n} \int_{|\lambda|>\frac{1}{8\mu_k}} \left|\left(\widehat{u}(t,\lambda)e_k, e_\ell\right)_{L^2(\mathbb{R}^n)}\right|^2 |\lambda|^n \,\mathrm{d}\lambda \\
&\doteq I^{\mathrm{low}} + I^{\mathrm{high}}.
\end{aligned}$$

Since

$$|\widehat{u}(t,\lambda)_{k,\ell}|^2 \lesssim \mathrm{e}^{-\delta t}\left(|\widehat{u}_0(\lambda)_{k,\ell}|^2 + |\widehat{u}_1(\lambda)_{k,\ell}|^2\right)$$

for any $|\lambda| > 1/(8\mu_k)$ and for some $\delta > 0$ (here δ and the unexpressed multiplicative constant hereafter are independent of the time variable and of the parameters λ and k, ℓ as well), by using Plancherel formula (this time for the Cauchy data u_0 and u_1) we find

$$I^{\mathrm{high}} \lesssim \mathrm{e}^{-\delta t}\left(\|u_0\|^2_{L^2(\mathbf{H}_n)} + \|u_1\|^2_{L^2(\mathbf{H}_n)}\right).$$

If we use again Plancherel formula to estimate I^{low} too, then, we get the $L^2 - L^2$ for $u(t,\cdot)$ without any decay factor in t, that is,

$$\|u(t,\cdot)\|^2_{L^2(\mathbf{H}_n)} \lesssim \|u_0\|^2_{L^2(\mathbf{H}_n)} + \|u_1\|^2_{L^2(\mathbf{H}_n)}$$

since for $|\lambda| < 1/(8\mu_k)$ we may only estimate

$$|\widehat{u}(t,\lambda)_{k,\ell}|^2 \lesssim \mathrm{e}^{-2\mu_k|\lambda|t} \left(|\widehat{u}_0(\lambda)_{k,\ell}|^2 + |\widehat{u}_1(\lambda)_{k,\ell}|^2\right)$$

If we want to improve this estimate we have to use $L^1(\mathbf{H}_n)$ regularity for initial data.

By Parseval's identity and by

$$\|\widehat{u}_h(\lambda)e_k\|_{L^2(\mathbb{R}^n)} \leq \|\widehat{u}_h(\lambda)\|_{\mathscr{L}(L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n))}\|e_k\|_{L^2(\mathbb{R}^n)} \leq \|u_h\|_{L^1(\mathbf{H}_n)}$$

for $h=0,1$, we get

$$\begin{aligned}
I^{\mathrm{low}} &\lesssim \sum_{k\in\mathbb{N}^n}\int_{0<|\lambda|<\frac{1}{8\mu_k}} \mathrm{e}^{-2\mu_k|\lambda|t}\left(\|\widehat{u}_0(\lambda)e_k\|^2_{L^2(\mathbb{R}^n)}+\|\widehat{u}_1(\lambda)e_k\|^2_{L^2(\mathbb{R}^n)}\right)|\lambda|^n\,\mathrm{d}\lambda\\
&\lesssim \sum_{k\in\mathbb{N}^n}\int_0^{\frac{1}{8\mu_k}} \mathrm{e}^{-2\mu_k\lambda t}\lambda^n\,\mathrm{d}\lambda\,\left(\|u_0\|^2_{L^1(\mathbf{H}_n)}+\|u_1\|^2_{L^1(\mathbf{H}_n)}\right)\\
&\lesssim \sum_{k\in\mathbb{N}^n}(2\mu_k t)^{-(n+1)}\int_0^{\frac{t}{4}} \mathrm{e}^{-\theta}\theta^n\,\mathrm{d}\theta\,\left(\|u_0\|^2_{L^1(\mathbf{H}_n)}+\|u_1\|^2_{L^1(\mathbf{H}_n)}\right)\\
&\lesssim t^{-\frac{\mathbb{Q}}{2}}\left(\|u_0\|^2_{L^1(\mathbf{H}_n)}+\|u_1\|^2_{L^1(\mathbf{H}_n)}\right),
\end{aligned}$$

in the last estimate we used that the series

$$\sum_{k\in\mathbb{N}^n}\mu_k^{-(n+1)} = \sum_{k\in\mathbb{N}^n}(2|k|+n)^{-(n+1)}$$

is convergent.

In order to estimate $\|u_t(t,\cdot)\|^2_{L^2(\mathbf{H}_n)}$ and $\|\nabla_{\mathrm{H}} u(t,\cdot)\|^2_{L^2(\mathbf{H}_n)}$ we can repeat similar computations since we know the representations of

$$\partial_t \widehat{u}(t,\lambda)_{k,\ell}$$

and of

$$\big((X_j u)^\wedge(t,\lambda)e_k, e_\ell\big)_{L^2(\mathbb{R}^n)} = \sqrt{\tfrac{|\lambda|}{2}}\big(\sqrt{k\cdot\epsilon_j}\,\widehat{u}(t,\lambda)_{k-\epsilon_j,\ell} - \widehat{u}(t,\lambda)_{k+\epsilon_j,\ell}\big),$$

$$\big((Y_j u)^\wedge(t,\lambda)e_k, e_\ell\big)_{L^2(\mathbb{R}^n)} = i\,\mathrm{sign}(\lambda)\sqrt{\tfrac{|\lambda|}{2}}\big(\sqrt{k\cdot\epsilon_j}\,\widehat{u}(t,\lambda)_{k-\epsilon_j,\ell} + \widehat{u}(t,\lambda)_{k+\epsilon_j,\ell}\big),$$

for any $k,\ell\in\mathbb{N}^n$, where $\{\epsilon_j\}_{1\leqslant j\leqslant n}$ is the canonical base of $\mathbb{R}^n$.

Let us point out that the relations for the Hermite functions

$$\partial_{w_j} e_k(w) = \frac{1}{\sqrt{2}}\left(\sqrt{k\cdot\epsilon_j}\; e_{k-\epsilon_j}(w) - e_{k+\epsilon_j}(w)\right)$$
$$w_j\, e_k(w) = \frac{1}{\sqrt{2}}\left(\sqrt{k\cdot\epsilon_j}\; e_{k-\epsilon_j}(w) + e_{k+\epsilon_j}(w)\right)$$

$(w\in\mathbb{R}^n)$ play a fundamental role in the determination of the decay estimate for the horizontal gradient of u.

Fischer V, Ruzhansky M,
Quantization on Nilpotent Lie Groups,
Progr. Math., vol. 314,
Birkhäuser/Springer, 2016.

Georgiev V, Palmieri A,
Lifespan estimates for local in time solutions to the semilinear heat equation on the Heisenberg group,
preprint,
arxiv:1905.05696 (2019).

Georgiev V, Palmieri A,
Critical exponent of Fujita-type for the semilinear damped wave equation on the Heisenberg group with power nonlinearity,
J. Differential Equations (2019), in press,
https://doi.org/10.1016/j.jde.2019.12.009

Ikehata R, Tanizawa K,
Global existence of solutions for semilinear damped wave equations in $\mathbb{R}^N$ with noncompactly supported initial data,
Nonlinear Anal. **61**(7) (2005), 1189-1208.

Matsumura A,
On the asymptotic behavior of solutions of semi-linear wave equations,
Publ. Res. Inst. Math. Sci. **12**(1) (1976/77), 169-189.

Palmieri A,
Decay estimates for the linear damped wave equation on the Heisenberg group,
preprint, arxiv:1908.02657 (2019).

Ruzhansky M, Tokmagambetov N,
Nonlinear damped wave equations for the sub-Laplacian on the Heisenberg group and for Rockland operators on graded Lie groups,
J. Differential Equations **265** (2018), 5212-5236.

Ruzhansky M, Yessirkegenov N,
Existence and non-existence of global solutions for semilinear heat equations and inequalities on sub-Riemannian manifolds, and Fujita exponent on unimodular Lie groups,
preprint,
arxiv:1812.01933v2 (2018).

Todorova G, Yordanov B,
Critical exponent for a nonlinear wave equation with damping,
J. Differential Equations **174**(2) (2001), 464-489.

Zhang QS,
A blow-up result for a nonlinear wave equation with damping: the critical case,
C. R. Acad. Sci. Paris Série I Math. **333**(2) (2001) 109–114.

Thank you for your attention!