Poincaré and Sobolev inequalities for differential forms in Heisenberg groups

B. Franchi (U. Bologna)
joint research with
A. Baldi (U. Bologna)
and
P. Pansu (U. Paris-Sud CNRS & U. Paris-Saclay)

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Let us start by considering the Euclidean setting $\mathbb{R}^N$.
If $1 \leq p < N$, we ask whether, given a closed differential h-form ω in $L^p(\mathbb{R}^N)$, there exists an $(h-1)$-form ϕ in $L^q(\mathbb{R}^N)$ for some $q \geq p$ such that $d\phi = \omega$ and

$$\|\phi\|_q \leq C \, \|\omega\|_p,$$

for $C = C(N, p, q, h)$. We refer to the above inequality as to the (p, q)-Poincaré inequality for h-forms (notice that, by the scale invariance, we must have $\frac{1}{p} - \frac{1}{q} = \frac{1}{N}$).

What is the connection between the above inequality and the classical Poincaré inequality? Classical Poincaré inquality for functions holds that, if $1 \leq p < N$, for any (say) Lipschitz continuous function u there exists a constant c_u such that

$$\|u - c_u\|_q \leq C(N,p)\,\|\nabla u\|_p \quad \text{provided} \quad \frac{1}{p} - \frac{1}{q} = \frac{1}{N}.$$

Classical Poincaré inequality for functions (i.e. 0-forms) can be derived from Poincaré inequality for differential forms as follows: we notice that $\omega := du$ is a closed form, so that, if there exists ϕ in $L^q(\mathbb{R}^N)$ such that $d\phi = \omega$, then $u - \phi = c_u$ (since $u - \phi$ is closed) and then

$$\|u - c_u\|_q = \|\phi\|_q \leq C\,\|du\|_p \leq C\,\|\nabla u\|_p.$$

We distinguish two different situations:

- "global" inequalities, i.e. inequalities on all $\mathbb{R}^N$;
- "local" inequalities in (suitable) bounded open subsets $\mathbb{R}^N$,

and, within each case, we must distinguish

- the case $1 < p < \infty$;
- the case $p = 1$.

Later on, we shall carry out the study of the same problems outside of the Euclidean setting (in fact, outside of the Riemannian setting) for the so-called Rumin's complex in Heisenberg groups.

Global inequalities: case $p > 1$.

If $p > 1$, the easy proof the Poincaré inequality on all $\mathbb{R}^N$ consists in putting $\phi = d^*\Delta^{-1}\omega$. Here, Δ^{-1} denotes the inverse of the Hodge Laplacian $\Delta = d^*d + dd^*$ and d^* is the formal L^2-adjoint of d. Indeed, if ω is a smooth compactly supported form, then

$$\omega = \Delta\Delta^{-1}\omega = (d^*d + dd^*)\Delta^{-1}\omega = d^*\Delta^{-1}d\omega + dd^*\Delta^{-1}\omega = d\phi.$$

The operator $d^*\Delta^{-1}$ is given by convolution with a kernel homogeneous of degree $1 - Q$, hence it is bounded from L^p to L^q if $p > 1$.

Unfortunately, this argument does not suffice for $p = 1$ since $d^*\Delta^{-1}$ maps L^1 only into the weak Marcinkiewicz space $L^{n/(n-1),\infty} \subset L^1_{\mathrm{loc}}$.

Case $p = 1$ and $h = 0$

It is true that upgrading from $L^{N/(N-1),\infty}$ to $L^{N/(N-1)}$ is possible for functions.
Indeed, for characteristic functions of sets, the $L^{N/(N-1),\infty}$ and $L^{N/(N-1)}$ norms coincide, and every function is the sum of characteristic functions of its superlevel sets.
In this way, we obtain the classical Gagliardo-Nirenberg inequality (in turn, equivalent to the isoperimetric inequality).
However, this trick does not seem to generalize to differential forms.

Case $p = 1$ and $h = N$

Poincaré inequality fails in degree n. There is an obvious obstruction: n-forms belonging to L^1 and with nonvanishing integral cannot be differentials of $L^{n/(n-1)}$ forms, (for a proof see [Pansu & Tripaldi, to appear]). But even if integral vanishes, a primitive ϕ such that $\|\phi\|_q \le C\,\|\omega\|_1$ need not exist, with $1 - \frac{1}{q} = \frac{1}{n}$. Indeed, if so, then, for every smooth function u on $\mathbb{R}^n$, one could write, for every n-form $\omega \in L^1$ with vanishing integral, (by Hahn-Banach theorem) that there exists a constant c_u such that $\|u - c_u\|_\infty \le C\,\|du\|_n$, that is impossible by an argument of capacity.

Case $p = 1$ and $h < N$

Surprisingly, Poincaré inequality persists for $p = 1$ in degree $h < N$.

Apparently, the integral obstruction still appears: if a closed L^1-form ω is the differential of a form in $L^{N/(N-1)}(\mathbb{R}^N)$, then for every constant coefficient form β of complementary degree, $\int \omega \wedge \beta = 0$.

Therefore we are lead to introduce the subspace L^1_0 of L^1-differential forms satisfying these conditions (we call them forms *with vanishing averages*).

However, in degree $h < N$, the obstruction is only apparent, thanks to the following result proved by Pansu & Tripadi:

Theorem

If $\omega \in L^1$ is a closed h-form of degree $h < N$, then $\omega \in L^1_0$.

Thus our Poincaré estimate reads as follows:

Let $h = 1, \dots, N-1$ *and set* $q = N/(N-1)$. *For every closed* h*-form* $\alpha \in L^1(\mathbb{R}^N)$ *there exists an* $(h-1)$*-form* $\phi \in L^q(\mathbb{R}^N)$, *such that*

$$d\phi = \alpha \qquad \textit{and} \qquad \|\phi\|_q \le C\,\|\alpha\|_1.$$

Furthermore, if α *is compactly supported, so is* ϕ.

The core of the proof consists of two points:

I) Lanzani-Stein inequality: *in degrees $h < N-1$, for smooth compactly supported forms ϕ,*

$$\|\phi\|_q \leq C\,(\|d\phi\|_1 + \|d^*\phi\|_1^*), \tag{1}$$

where $\|\cdot\|_1^*$ denotes either L^1-norm (in degrees $h \neq 1$) or the norm of the real Hardy space $\mathcal{H}^1$ (in degree $h = 1$). Moreover, if $h = N-1$, then

$$\|\phi\|_q \leq C\,(\|d\phi\|_1^* + \|d^*\phi\|_1). \tag{2}$$

In other words, the classical Gagliardo-Nirenberg inequality is the first link of a chain of analogous inequalities for compactly supported smooth differential forms.

Mimicking the proof for $p > 1$ one could be tempted to plug $\phi = d^*\Delta^{-1}\alpha$ in Lanzani-Stein inequality to prove Poincaré inequality, replacing the usual $L^p - L^q$ boundedness of singular integrals of potential type by Lanzani-Stein inequality. Unfortunately, this is not possible, since ϕ only belongs to L^1_{loc} (in particular, is not compactly supported).

It is natural to expect that this difficulty can be bypassed by means of truncation and regularization arguments. However, this approach produces commutation terms that have to be handled carefully.

If $N \in \mathbb{N}$, let now χ_N be a cut-off function supported in $B(0,2N)$, $\chi_N \equiv 1$ on $B(0,N)$. If $\varepsilon < 1$ let J_ϵ be an usual Friedrichs' mollifier. Then, set

$$v_{\varepsilon,N} := J_\varepsilon * d^*(\chi_N \Delta_{\mathbb{H},h}^{-1}\omega),$$

and plug $v_{\varepsilon,N}$ in Lanzani-Stein inequality.
Obviously,

$$d^* v_{\varepsilon,N} = 0.$$

We get

$$\begin{aligned}
\|v_{\varepsilon,N}\|_{L^{N/(N-1)}} &\le C\|dv_{\varepsilon,N}\|_{L^1(\mathbb{H}^n)} = C\|J_\varepsilon * dd^*(\chi_N\Delta^{-1}\omega)\|_{L^1(\mathbb{H}^n)} \\
&\le C\big\{\|J_\varepsilon * [dd^*,\chi_N](\Delta^{-1}\omega)\|_{L^1} + \|J_\varepsilon * \chi_N(dd^*\Delta^{-1}\omega)\|_{L^1}\big\} \\
&\le C\big\{\|[dd^*,\chi_N](\Delta^{-1}\omega)\|_{L^1} + \|\chi_N(dd^*\Delta^{-1}\omega)\|_{L^1}\big\}.
\end{aligned}$$

By duality, we can show that

$$\begin{aligned}\omega = \Delta\Delta^{-1}\omega &= dd^*\Delta^{-1}\omega + d^*d\Delta^{-1}\omega \\ &= dd^*\Delta^{-1}\omega + d^*\Delta^{-1}d\omega = dd^*\Delta^{-1}\omega.\end{aligned}$$

Thus, the point is to estimate the garbage term $\|[dd^*, \chi_N]\Delta^{-1}\omega\|_1$ when $N \to \infty$.

Notice that $[dd^*, \chi_N]$ is a first order differential operator, of the form $[dd^*, \chi_N] = P_0 + P_1$ where P_0 has order 0 and depends on second derivatives $\nabla^2 \chi_N$ and P_1 has order 1 and depends on first derivatives $\nabla \chi_N$ only, so they are both supported in shells $N < |x| < 2N$.

II) We come now to our basic trick: If P is the operator of convolution with a homogeneous kernel of degree $\mu - N$ with $\mu > 0$, and $\alpha \in L^1$, then it is not difficult to prove that the L^1 norm of $P\alpha$ on shells $B(0, 2N) \setminus B(0, N)$ is $O(R^\mu)$.

If furthermore $\alpha \in L^1_0$, this can be improved to $o(R^\mu)$.
Keeping in mind that all components of ω have vanishing average, we obtain eventually

$$\|[dd^*, \chi_N]\Delta^{-1}\omega\|_1 \to 0 \qquad \text{as } N \to \infty.$$

This makes possible to take $u := d^*(\chi_N \Delta^{-1}\alpha)$ in Lanzani-Stein inequality, getting rid of the commutators.

Local inequalities: case $p > 1$.

Consider now the the local inequalities. If $p > 1$ then Iwaniec & Lutoborski proved that a Poincaré inequality holds in convex set, and Mitrea, Mitrea & Monniaux proved that, in star-shaped Lipschitz domains, Poincaré inequality holds and, in addition, if ω is compactly supported, so is ϕ (Sobolev inequality).

Local inequalities: case $p = 1$.

Unfortunately, Iwaniec & Lutoborski's argument cannot cover the case $p = 1$. Suppose indeed D is a bounded convex open set. The core of Iwaniec & Lutoborski's argument relies on the construction of an homotopy operator $T_{IL} : L^p(D) \to W^{1,p}(D)$ which is defined by a kernel k that can be estimated by a singular integral of potential type that is homogeneous of degree $1 - N$

We can prove a local versions of Poincaré inequality, for $p = 1$, of the following type.
We call it an *interior* Poincaré inequality, or a Poincaré inequality *loss of the domain* since the L^q norm of ϕ in a ball $B(0, R)$ is controlled by the L^1-norm of ω in a slightly larger ball $B(0, \lambda R)$ with $\lambda > 1$.

We have:

For $h = 1, \ldots, n-1$, let $q = N/(N-1)$. For every $\lambda > 1$, there exists C with the following property. Let $B(R)$ be a ball of radius R in $\mathbb{R}^N$. Then the following "interior Poincaré inequality" holds: for every closed h-form $\omega \in L^1(B(\lambda R))$, there exists an $(h-1)$-form $\phi \in L^q(B(R))$, such that

$$d\phi = \omega_{|B(R)} \qquad \textit{and} \qquad \|\phi\|_{L^q(B(R))} \leq C \, \|\omega\|_{L^1(B(\lambda R))}.$$

Let us give a sketch of the proof.

Starting from the homotopy operator $d^*\Delta^{-1}$ (that is associated with an homogeneous kernel), through successive localizations obtained by means of a family of cut-off functions, we obtain an approximate homotopy formula for L^1-forms α on $B(\lambda R)$ such that $d\alpha \in L^1(\lambda R)$:

$$\alpha = dT\alpha + Td\alpha + S\alpha \qquad \text{on } B(R)$$

(here is the loss of the domain).

Here T a bounded operator

$$T : L^1(B(\lambda R)) \cap d^{-1}(L^1(B(\lambda R))) \to L^q(B(R))$$

(in other words, T has the good continuity properties), and S is a smoothing operator

$$S : L^1(B(\lambda R)) \to W^{s,q}(B(R)).$$

Take now $\alpha = \omega$ that is a closed form. Thus $S\omega = \omega - dT\omega$ is closed and belongs to $L^q(B(R))$, with norm controlled by the L^1-norm of ω in $B(\lambda R)$). Thus we can apply Iwaniec & Lutoborski's homotopy T_{IL} to obtain

$$S\omega = dT_{IL}S\omega =: d\gamma$$

on $B(R)$ with the norm of γ in $W^{1,q}(B(R))$ controlled by the L^q-norm of $S\omega$ in $B(R)$, and therefore by the L^1-norm of ω. Set $\phi := T\alpha + \gamma$. Clearly

$$d\phi = dT\omega + d\gamma = dT\omega + S\omega = \omega.$$

In addition

$$\|\phi\|_{L^q(B(R)} \leq C\big(\|\omega\|_{L^1(B(R))} + \|S\omega\|_{L^q(B(R)}\big) \leq C\|\omega\|_{L^1(B(R))}.$$

The Heisenberg groups setting

We denote by $\mathbb{H}^n$ the n-dimensional Heisenberg group identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted by $p = (x, y, t)$, with both $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. If p and $p' \in \mathbb{H}^n$, the group operation is defined as

$$p \cdot p' = (x + x', y + y', t + t' + \frac{1}{2}\sum_{j=1}^{n}(x_j y_j' - y_j x_j')).$$

We denote by $\mathfrak{h}$ the Lie algebra of the left invariant vector fields of $\mathbb{H}^n$.
As customary, $\mathfrak{h}$ is identified with the tangent space $T_e\mathbb{H}^n$ at the origin.

The standard basis of $\mathfrak{h}$ is given, for $i = 1, \dots, n$, by

$$X_i := \partial_{x_i} - \frac{1}{2} y_i \partial_t, \quad Y_i := \partial_{y_i} + \frac{1}{2} x_i \partial_t, \quad T := \partial_t.$$

Throughout this talk, to avoid cumbersome notations, we write also

$$W_i := X_i, \quad W_{i+n} := Y_i, \quad W_{2n+1} := T, \quad \text{for } i = 1, \cdots, n. \tag{3}$$

The only non-trivial commutation relations are $[X_j, Y_j] = T$, for $j = 1, \ldots, n$.

- The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$.

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- The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$.
- Denoting by $\mathfrak{h}_2$ the linear span of T, the 2-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

If $\lambda > 0$, the stratification of the Lie algebra $\mathfrak{h}$ induces a family of non-isotropic dilations δ_λ, $\lambda > 0$ in $\mathbb{H}^n$ (automorphisms of the group) as follows:

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t).$$

The differential operators X_j, Y_j are homogeneous of degree 1 with respect to δ_λ, whereas the operator T is homogeneous of degree 2.

The Lebesgue measure $\mathcal{L}^{2n+1}$ in $\mathbb{H}^n$ is invariant under group translations and homogenous of degree

$$Q := 2n + 2$$

with respect to group dilations. We call Q the homogeneous dimension of $\mathbb{H}^n$.

Heisenberg groups carry an intrinsic structure of metric space associated with the family of balls (Korányi balls) defined by

$$B(e,r) = \{|x|^4 + |y|^4 + t^2 \le r^4\}$$

and

$$B(p,r) = p \cdot B(e,r).$$

The vector space $\mathfrak{h}$ can be endowed with an inner product, indicated by $\langle \cdot, \cdot \rangle$, making $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ and T orthonormal.

The dual space of $\mathfrak{h}$ is denoted by $\bigwedge^1 \mathfrak{h}$. The basis of $\bigwedge^1 \mathfrak{h}$, dual to the basis $\{X_1, \dots, Y_n, T\}$, is the family of covectors $\{dx_1, \dots, dx_n, dy_1, \dots, dy_n, \theta\}$ where

$$\theta := dt - \frac{1}{2} \sum_{j=1}^{n} (x_j dy_j - y_j dx_j)$$

is called the *contact form* in $\mathbb{H}^n$.

If we denote by $\bigwedge^1 \mathfrak{h}_1$ the span of $\{dx_1, \dots, dx_n, dy_1, \dots, dy_n\}$, (the horizontal 1-forms) we can write

$$\bigwedge^1 \mathfrak{h} = \bigwedge^1 \mathfrak{h}_1 \oplus \operatorname{span} \theta.$$

Starting from $\bigwedge^1 \mathfrak{h}$ we can define the space $\bigwedge^k \mathfrak{h}$ of the k-covectors. Analogously, starting from $\bigwedge^1 \mathfrak{h}_1$ we can define the space $\bigwedge^k \mathfrak{h}_1$ of the horizontal k-covectors and we have:

$$\bigwedge^k \mathfrak{h} = \bigwedge^k \mathfrak{h}_1 \oplus \theta \wedge \bigwedge^{h-1} \mathfrak{h}_1 .$$

In other words, unlike in the Euclidean setting, the space of k-forms splits into two subspaces of homogeneity k and $k+1$, respectively (we shall refer to the homogeneity degree of a form as to its "weight"). This makes de Rham complex not "natural" for the group structure.

Rumin's complex

Rumin's complex $(E_0^\bullet, d_c)$ is meant to overcome this difficulty, still preserving the cohomology of de Rham's complex.

We refer to [Rumin, *J. Differential Geom.* 39 (1994)], [Rumin, *Rend. Circ. Mat. Palermo* (2005)], [Baldi, F., Tchou & Tesi, *Adv. Math.* (2010)], [F. & Tripaldi, *Unione Matematica Italiana* (2015)] for details of the construction.

In the present talk, we shall merely need the following list of formal properties.

- For $h = 0, \ldots, 2n+1$, the space of Rumin h-forms, E_0^h is the space of smooth sections of a left-invariant subbundle of $\bigwedge^h \mathfrak{h}$ (that we still denote by E_0^h). Hence it inherits inner products, L^p and $W^{s,p}$ norms.

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- For $h = 0, \dots, 2n+1$, the space of Rumin h-forms, E_0^h is the space of smooth sections of a left-invariant subbundle of $\bigwedge^h \mathfrak{h}$ (that we still denote by E_0^h). Hence it inherits inner products, L^p and $W^{s,p}$ norms.
- A differential operator $d_c : E_0^h \to E_0^{h+1}$ is defined. It is left-invariant, homogeneous with respect to group dilations. It is a first order homogeneous operator in the horizontal derivatives in degree $\neq n$, whereas *it is a second order homogeneous horizontal operator in degree n.*

- The L^2 (formal) adjoint of d_c is a differential operator d_c^* of the same order as d_c.

- The L^2 (formal) adjoint of d_c is a differential operator d_c^* of the same order as d_c.
- Hypoelliptic "Laplacians" can be formed from d_c and d_c^*.
- Altogether, operators d_c form a complex: $d_c \circ d_c = 0$.

This complex is homotopic to de Rham's complex $(\Omega^\bullet, d)$. The homotopy is achieved by differential operators $\Pi_E : E_0^\bullet \to \Omega^\bullet$ and $\Pi_{E_0} : \Omega^\bullet \to E_0^\bullet$ (Π_E has horizontal order ≤ 1 and Π_{E_0} is an algebraic operator).

In other words, $\Pi_E : E_0^\bullet \to \Omega^\bullet$ and $\Pi_{E_0} : \Omega^\bullet \to E_0^\bullet$ intertwine differentials d_c and d,

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d_c} & E_0^h & \xrightarrow{d_c} & E_0^{h+1} & \xrightarrow{d_c} & \cdots \\
 & & \Pi_E \big\downarrow & & \Pi_E \big\downarrow & & \\
\cdots & \xrightarrow{d} & \Omega^h & \xrightarrow{d} & \Omega^{h+1} & \xrightarrow{d} & \cdots
\end{array}$$

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d_c} & E_0^h & \xrightarrow{d_c} & E_0^{h+1} & \xrightarrow{d_c} & \cdots \\
 & & \big\uparrow \Pi_{E_0} & & \big\uparrow \Pi_{E_0} & & \\
\cdots & \xrightarrow{d} & \Omega^h & \xrightarrow{d} & \Omega^{h+1} & \xrightarrow{d} & \cdots
\end{array}$$

and there exists an algebraic operator $A : \Omega^\bullet \to \Omega^{\bullet-1}$ such that $1 - \Pi_{E_0}\Pi_E\Pi_E\Pi_{E_0} = 0$ on $E_0^\bullet$ and $1 - \Pi_E\Pi_{E_0}\Pi_{E_0}\Pi_E = dA + Ad$ on $\Omega^\bullet$.

In $\mathbb{H}^n$, following Rumin, we define the operator $\Delta_{\mathbb{H},h}$ on E_0^h by setting

$$\Delta_{\mathbb{H},h} = \begin{cases} d_c\delta_c + \delta_c d_c & \text{if} \quad h \neq n, n+1; \\ (d_c\delta_c)^2 + \delta_c d_c & \text{if} \quad h = n; \\ d_c\delta_c + (\delta_c d_c)^2 & \text{if} \quad h = n+1. \end{cases}$$

For sake of simplicity, since a basis of E_0^h is fixed, the operator $\Delta_{\mathbb{H},h}$ can be identified with a matrix-valued map, still denoted by $\Delta_{\mathbb{H},h}$

$$\Delta_{\mathbb{H},h} = (\Delta_{\mathbb{H},h}^{ij})_{i,j=1,\dots,N_h} : \mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h}) \to \mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h}), \tag{4}$$

Theorem [Baldi, F., Tesi] *If* $0 \le h \le 2n+1$, *then the differential operator* $\Delta_{\mathbb{H},h}$ *is homogeneous of degree* a *with respect to group dilations, where* $a = 2$ *if* $h \neq n, n+1$ *and* $a = 4$ *if* $h = n, n+1$.
It follows that
i) for $j = 1, \dots, N_h$ *there exists*

$$K_j = \big(K_{1j}, \dots, K_{N_h j}\big), \quad j = 1, \dots N_h \tag{5}$$

with $K_{ij} \in \mathcal{D}'(\mathbb{H}^n) \cap \mathcal{E}(\mathbb{H}^n \setminus \{0\})$, $i, j = 1, \dots, N$;
ii) if $a < Q$, *then the* K_{ij}*'s are kernels of type* a *for*
$i, j = 1, \dots, N_h$
If $a = Q$, *then the* K_{ij}*'s satisfy the logarithmic estimate* $|K_{ij}(p)| \le C(1 + |\ln \rho(p)|)$ *and hence belong to* $L^1_{\mathrm{loc}}(\mathbb{H}^n)$.
Moreover, their horizontal derivatives $W_\ell K_{ij}$, $\ell = 1, \dots, 2n$, *are kernels of type* $Q - 1$;

iii) when $\alpha \in \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h})$, if we set

$$\Delta_{\mathbb{H},h}^{-1}\alpha := \Big(\sum_j \alpha_j * K_{1j}, \dots, \sum_j \alpha_j * K_{N_h j}\Big), \tag{6}$$

then $\Delta_h \Delta_{\mathbb{H},h}^{-1}\alpha = \alpha$. Moreover, if $a < Q$, also $\Delta_{\mathbb{H},h}^{-1}\Delta_h\alpha = \alpha$.
iv) if $a = Q$, then for any $\alpha \in \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h})$ there exists $\beta_\alpha := (\beta_1, \dots, \beta_{N_h}) \in \mathbb{R}^{N_h}$, such that

$$\Delta_{\mathbb{H},h}^{-1}\Delta_h\alpha - \alpha = \beta_\alpha.$$

v) when $\alpha \in \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h})$, if we set

$$\mathcal{K}\alpha := \Big(\sum_j \alpha_j * K_{1j}, \dots, \sum_j \alpha_j * K_{N_h j}\Big), \tag{7}$$

then $\Delta_{\mathbb{H},h}\mathcal{K}\alpha = \alpha$. Moreover, if $a < Q$, also $\mathcal{K}\Delta_{\mathbb{H},h}\alpha = \alpha$.

vi) if $a = Q$, then for any $\alpha \in \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h})$ there exists $\beta_\alpha := (\beta_1, \dots, \beta_{N_h}) \in \mathbb{R}^{N_h}$, such that

$$\mathcal{K}\Delta_{\mathbb{H},h}\alpha - \alpha = \beta_\alpha.$$

The operator $\mathcal{K}$ can be identified with an operator (still denoted by $\mathcal{K}$) acting on smooth compactly supported differential forms in $\mathcal{D}(\mathbb{H}^n, E_0^h)$. Moreover, when the notation will not be misleading, we shall denote by $\alpha \to \Delta_{\mathbb{H},h}^{-1}\alpha$ the convolution with $\mathcal{K}$ acting on forms of degree h.

Global inequalities in $\mathbb{H}^n$: case $p > 1$.

Global Poincaré and Sobolev inequalities in Heisenberg groups when $p > 1$ can be easily proved mimicking verbatim the proof of the corresponding results in the Euclidean setting, using the estimates of the fundamental solution of $\Delta_{\mathbb{H},h}$ and the continuity properties in L^p-spaces for singular integrals of potential type in Carnot groups (see [Folland] and [Folland & Stein]).

Let $h = 1, \dots, 2n$ *and* $1 < p < Q$ *if* $h \neq n+1$ *and* $1 < p < Q/2$ *if* $h = n+1$*. Set* $q = pQ/(Q-p))$ *if* $h \neq n+1$ *and* $q = Q/(Q-2p)$ *if* $h = n+1$*. For every* d_c*-closed* h*-form* $\alpha \in L^1(\mathbb{H}^n)$*, there exists an* $(h-1)$*-form* $\phi \in L^q(\mathbb{H}^n)$*, such that*

$$d_c\phi = \alpha \qquad \text{and} \qquad \|\phi\|_q \leq C\,\|\alpha\|_1.$$

Furthermore, in both cases, if α *is compactly supported, so is* ϕ*.*

Remark Again the assertion fails to hold in top degree $h = 2n + 1$.

Local inequalities in $\mathbb{H}^n$: case $p > 1$.

The local inequality requires a new argument because of the loss of an "intrinsic" Iwaniec & Lutoborski homotopy formula. This difficulty can be bypassed by combining Iwaniec & Lutoborski homotopy formula with the homotopy Π_E. The $L^p - L^q$ estimates can be proved by a suitable iteration argument.

Global and local inequalities in $\mathbb{H}^n$: case $p = 1$.

The following global Poincaré inequality holds in $\mathbb{H}^n$.
Let $h = 1, \dots, 2n$ and set $q = Q/(Q-1))$ if $h \neq n+1$ and $q = Q/(Q-2)$ if $h = n+1$. For every d_c-closed h-form $\alpha \in L^1(\mathbb{H}^n)$, there exists an $(h-1)$-form $\phi \in L^q(\mathbb{H}^n)$, such that

$$d_c\phi = \alpha \qquad \text{and} \qquad \|\phi\|_q \leq C\,\|\alpha\|_1.$$

Furthermore, in both cases, if α is compactly supported, so is ϕ.

In turn, this global Poincaré inequality has a local counterpart that reads as follows:

If $h = 1, \dots, 2n$, let $q = (2n+2)/(2n+1)$ when $h \neq n+1$ and $q = (2n+2)/(2n)$ when $h = n+1$. There exist $\lambda > 1$ and C with the following property. Let $B(R)$ be a ball of radius R in $\mathbb{H}^n$. Then for every d_c-closed Rumin h-form $\alpha \in L^1(B(\lambda R))$, there exists an $(h-1)$-form $\phi \in L^q(B(R))$, such that

$$d_c\phi = \alpha_{|B(R)} \qquad \textit{and} \qquad \|\phi\|_{L^q(B(R))} \leq C\,\|\alpha\|_{L^1(B(\lambda R))}.$$

We try to mimic the proof of the Euclidean case.
First of all we notice that a suitable form of Lanzani-Stein inequality holds within Rumin's complex (a suitable form since we have to keep into account the orders of d_c and δ_c that can equal 2 in some cases).
We refer to [Baldi & F. 2013] and [Baldi, F. & Pansu, 2016].

We use Rumin's Laplacian $\Delta_{\mathbb{H}}$ on Rumin forms. It does not quite commute with Rumin's differential d_c in degrees $n-1$ and $n+2$ but this turns out to be harmless. Write $K = d_c^* \Delta_{\mathbb{H}}^{-1}$ (with a modification in degrees n and $n+1$), in order that $d_c K + K d_c = 1$ on smooth compactly supported forms. In spite of the complicated form of Leibniz' formula for d_c, the basic features of commutators $[d_c d_c^*, \chi]\Delta_{\mathbb{H}}^{-1}$ from the Euclidean case persist.

The local Poincaré inequality requires special care in the Heisenberg case, since no analogue of Iwaniec-Lutoborsky's homotopy exists. The kernel of $K = d_c^* \Delta_{\mathbb{H}}^{-1}$ is a valuable replacement. This provides again a L^1 local primitive for a d_c-closed form, up to a smoothed d_c-closed form, which belongs to $W^{3,1}$.

The L^1 primitive is upgraded to L^q using a cut-off and our global inequality in the same manner. To the smoothed form, one can apply Rumin's homotopy, yielding a $W^{2,1}$ d_c-closed form, and then Iwaniec-Lutoborsky's Euclidean homotopy. The resulting form belongs to L^q, with $q = (2n+2)/(2n+1)$ if $h \neq n+1$ and $q = (2n+2)/(2n)$ if $h = n+1$, again by Sobolev embedding.