Heat and entropy flows in Carnot groups

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Some topics of Geometric Analysis and Geometric Measure Theory

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Sketchy idea: Otto's calculus

Take $(v_t)_{t>0} \colon \mathbb{R}^n \to \mathbb{R}^n$ time-dependent vector field. Consider the solution of

$$\partial_t u_t + \mathrm{div}(v_t u_t) = 0 \qquad \text{in } \mathbb{R}^n \times (0, +\infty). \tag{CE}$$

In Otto's calculus, v_t is the "velocity" of $\mu_t = u_t \mathscr{L}^n \in \mathscr{P}(\mathbb{R}^n)$ for $t > 0$.

The (Shannon-Boltzmann) entropy

$$\mathrm{Ent}(\mu_t) := \int_{\mathbb{R}^n} u_t \log u_t \, dx$$

along the curve $(\mu_t)_{t>0}$ satisfies

$$\frac{d}{dt}\mathrm{Ent}(\mu_t) \overset{\text{(CE)}}{=} -\int_{\mathbb{R}^n} (1 + \log u_t) \, \mathrm{div}(v_t u_t) \, dx = \int_{\mathbb{R}^n} \left\langle \frac{\nabla u_t}{u_t}, v_t \right\rangle d\mu_t.$$

Then $(\mu_t)_{t>0}$ is a gradient flow of Ent if $t \mapsto \mathrm{Ent}(\mu_t)$ has maximal dissipation rate.

Having in mind the formal differentiation "$\frac{d}{dt}\mathrm{Ent}(\mu_t) = \nabla \mathrm{Ent}(\mu_t) \cdot v_t$", we deduce

$$(\mu_t)_{t>0} \text{ gradient flow of Ent} \iff v_t = -\frac{\nabla u_t}{u_t} \iff \partial_t u_t = \Delta u_t.$$

Problem: prove this equivalence (rigorously!) in a metric measure space $(X, \mathsf{d}, \mathfrak{m})$.

Gradient flows in (X, d)

Let (X, d) be a metric space. A curve $\gamma\colon I \subset \mathbb{R} \to X$ is $AC^p(I; (X, \mathsf{d}))$ if

$$\exists g \in L^p(I) \qquad \text{such that} \qquad \mathsf{d}(\gamma_s, \gamma_t) \le \int_s^t g(r)\, dr \qquad \forall s, t \in I,\ s < t.$$

The minimal $g \in L^p(I)$ is the metric derivative $|\dot\gamma_t| = \lim_{s \to t} \frac{\mathsf{d}(\gamma_s, \gamma_t)}{|s-t|}$.

Let $E\colon X \to \mathbb{R} \cup \{+\infty\}$ be a function. A metric gradient flow of E starting from $\gamma_0 \in \mathrm{Dom}(E)$ is a curve $\gamma \in AC^1_{\mathrm{loc}}([0, +\infty); (X, \mathsf{d}))$ such that

$$E(\gamma_t) + \frac{1}{2}\int_s^t |\dot\gamma_r|^2\, dr + \frac{1}{2}\int_s^t |\mathrm{D}^- E|^2(\gamma_r)\, dr \le E(\gamma_s) \qquad \forall t > s \ge 0. \quad \text{(EDI)}$$

Here $|\mathrm{D}^- E|(x) = \limsup_{y \to x} \max\left\{ \frac{E(x) - E(y)}{\mathsf{d}(x,y)}, 0 \right\}$ is the descending slope of E.

If X is a Hilbert space, then $\dot\gamma_t = -\nabla E(\gamma_t)$ and so

$$\frac{d}{dt} E(\gamma_t) \overset{\text{(chain)}}{=} \langle \nabla E(\gamma_t), \dot\gamma_t \rangle \overset{\text{(CS)}}{=} -|\nabla E(\gamma_t)| \cdot |\dot\gamma_t| \overset{\text{(Y)}}{=} -\frac{1}{2}|\nabla E(\gamma_t)|^2 - \frac{1}{2}|\dot\gamma_t|^2.$$

Entropy flows in $(\mathscr{P}_2(X), W_2)$

Let (X, d) be a Polish (geodesic) metric space. We endow the set

$$\mathscr{P}_2(X) = \left\{ \mu \in \mathscr{P}(X) : \int_X \mathsf{d}(x, x_0)^2 \, d\mu(x) < +\infty, \; x_0 \in X \right\}$$

with the Wasserstein distance: for any $\mu, \nu \in \mathscr{P}(X)$, we set

$$W_2^2(\mu, \nu) = \inf \left\{ \int_{X \times X} \mathsf{d}(x, y)^2 \, d\pi : \pi \in \Gamma(\mu, \nu) \right\},$$

where $\Gamma(\mu, \nu) = \{\pi \in \mathscr{P}(X \times X) : (p_1)_\# \pi = \mu, \; (p_2)_\# \pi = \nu\}$.

Property: $(\mathscr{P}_2(X), W_2)$ is a Polish (geodesic) metric space.

Let $\mathfrak{m}$ be a non-negative, σ-finite Borel measure on X such that

$$\mathfrak{m}\left(\{x \in X : d(x, x_0) < r\}\right) \leq A e^{B r^2} \qquad \exists A, B > 0. \tag{exp.ball}$$

The entropy functional $\mathrm{Ent}_{\mathfrak{m}} : \mathscr{P}_2(X) \to (-\infty, +\infty]$ is defined as

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) = \begin{cases} \displaystyle\int_X \varrho \log \varrho \, d\mathfrak{m} & \text{if } \mu = \varrho \mathfrak{m} \in \mathscr{P}_2(X), \\ +\infty & \text{otherwise.} \end{cases}$$

Assumption (exp.ball) ensures that $\mathrm{Ent}(\mu) > -\infty$ for all $\mu \in \mathscr{P}_2(X)$.

Heat flows in $(X, \mathsf{d}, \mathfrak{m})$

Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. The Cheeger energy of a function $u\colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is given by

$$\mathsf{Ch}(u) = \inf\left\{\liminf_{n} \int_X |\mathsf{D}u_n|^2 \, d\mathfrak{m} : u_n \to u \text{ in } L^2(X, \mathfrak{m}),\ u_n \in \mathsf{Lip}(X; \mathbb{R})\right\}.$$

Here $|\mathsf{D}u|(x) = \limsup_{y \to x} \frac{|u(y)-u(x)|}{\mathsf{d}(x,y)}$ denotes the slope of $u \in \mathsf{Lip}(X; \mathbb{R})$.

Properties: Cheeger energy is convex, l.s.c. and $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is dense.

The heat flow in $(X, \mathsf{d}, \mathfrak{m})$ is the (Hilbertian) gradient flow of Ch in $L^2(X, \mathfrak{m})$: for $u_0 \in L^2(X, \mathfrak{m})$, $\exists t \mapsto u_t = \mathsf{H}_t(u_0) \in \mathsf{Lip}_{\mathsf{loc}}((0, +\infty); L^2(X, \mathfrak{m}))$ such that

$$u_t \xrightarrow[t \to 0]{} u_0 \text{ in } L^2(X, \mathfrak{m}) \qquad \text{and} \qquad \frac{d}{dt} u_t \in -\partial^- \mathsf{Ch}(u_t) \text{ for a.e. } t > 0.$$

We set $-\Delta u \in \partial^- \mathsf{Ch}(u)$ the element of minimal $L^2(X, \mathfrak{m})$-norm.

Be careful: $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ with $\|u\|_{W^{1,2}} = \left(\|u\|_{L^2}^2 + \mathsf{Ch}(u)\right)^{1/2}$ is Banach, but not Hilbert in general! For example, consider $(\mathbb{R}^n, \|\cdot\|_p, \mathscr{L}^n)$ for $p \neq 2$.

$CD(K,+\infty)$ metric measure spaces

The space $(X,\mathsf{d},\mathfrak{m})$ is $CD(K,+\infty)$ for some $K\in\mathbb{R}$ if $\forall\mu_0,\mu_1\in$ Dom(Ent) $\exists\mu_t\colon[0,1]\to\mathscr{P}_2(X)$ constant speed geodesic joining μ_0 and μ_1 such that

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_t)\le(1-t)\mathrm{Ent}_{\mathfrak{m}}(\mu_0)+t\,\mathrm{Ent}_{\mathfrak{m}}(\mu_1)-\frac{K}{2}t(1-t)W_2^2(\mu_0,\mu_1). \qquad \text{(K)}$$

The space $(X,\mathsf{d},\mathfrak{m})$ is $RCD(K,+\infty)$ if, in addition, $W^{1,2}(X,\mathsf{d},\mathfrak{m})$ is Hilbertian.

Theorem (von Renesse - Sturm, 2005)

Let (M,g) be a Riemannian manifold. Then (K) holds if and only if $Ric\ge K$.

Theorem (Gigli, 2010; Ambrosio - Gigli - Savaré, 2014)

Assume $(X,\mathsf{d},\mathfrak{m})$ is (exp.ball) and $CD(K,+\infty)$. Consider

- $u_t=\mathrm{H}_t(u_0)$ a GF of Ch in $L^2(X,\mathfrak{m})$ starting from $u_0\in L^2(X,\mathfrak{m})$;
- μ_t a GF of Ent in $(\mathscr{P}_2(X),W_2)$ starting from $\mu_0=u_0\mathfrak{m}\in$ Dom(Ent).

Then the two GFs are unique and coincide, i.e. $\mu_t=u_t\mathfrak{m}$ for all $t>0$.

- X Riemannian manifold with $Ric\ge K$ (Erbar, 2010);
- X Alexandrov space (Gigli - Kuwada - Otha, 2009 - 2013).

Non-$CD(K, +\infty)$ spaces: Carnot groups

A Carnot group $\mathbb{G}$ is a connected, simply connected, stratified Lie group with

$$\mathrm{Lie}(\mathbb{G}) = V_1 \oplus V_2 \oplus \cdots \oplus V_\kappa, \quad V_i = [V_1, V_{i-1}], \quad [V_1, V_\kappa] = \{0\}.$$

By Campbell-Hausdorff formula, $\mathbb{G} \sim (\mathbb{R}^n, \cdot)$ using exponential coordinates.

We call $H\mathbb{G} = V_1$ the horizontal directions. If $V_1 = \mathrm{span}\{X_1, \dots, X_m\}$, then $\nabla_{\mathbb{G}} u = \sum_{j=1}^m (X_j u) X_j \in V_1$ and $\Delta_{\mathbb{G}} u = \sum_{j=1}^m X_j^2 u$ (Kohn's sub-Laplacian).

The Carnot-Carathéodory distance of $x, y \in \mathbb{G}$ is

$$\mathsf{d}_{\mathsf{cc}}(x, y) = \inf\left\{ \int_0^1 \|\dot\gamma_t\|_{\mathbb{G}}\, dt : \gamma \in \mathrm{Lip}([0,1]; \mathbb{R}^n),\ \gamma_0 = x,\ \gamma_1 = y,\ \dot\gamma_t \in V_1 \right\}.$$

Then $(\mathbb{G}, \mathsf{d}_{\mathsf{cc}}, \mathscr{L}^n)$ is Polish, geodesic and $\mathscr{L}^n(\mathsf{B}_{\mathsf{cc}}(x, r)) = C r^Q$, $Q \in \mathbb{N}$.

Proposition (Ambrosio - S., 2018)

The space $(\mathbb{G}, \mathsf{d}_{\mathsf{cc}}, \mathscr{L}^n)$ is not $CD(K, +\infty)$ for any $K \in \mathbb{R}$!

Correspondence of the two GFs in Carnot groups

Theorem (Juillet, 2014)

The space $(\mathbb{H}^n, \mathsf{d}_{cc}, \mathscr{L}^{2n+1})$ is not $CD(K, +\infty)$ [Juillet, 2009]. Consider

- u_t solution of $\partial_t u_t = \Delta_{\mathbb{H}^n} u_t$ with initial datum $u_0 \in L^1(\mathbb{H}^n, \mathscr{L}^{2n+1})$;
- μ_t a GF of Ent in $(\mathscr{P}_2(\mathbb{H}^n), W_2)$ from $\mu_0 = u_0 \mathscr{L}^{2n+1} \in \mathsf{Dom}(\mathsf{Ent})$.

Then the two GFs are unique and coincide, i.e. $\mu_t = u_t \mathscr{L}^{2n+1}$ for all $t > 0$.

Problem [Juillet, 2014]: is it true in any Carnot group? YES!

Theorem (Ambrosio - S., 2018)

Let $(\mathbb{G}, \mathsf{d}_{cc}, \mathscr{L}^n)$ be a Carnot group. Consider

- u_t solution of $\partial_t u_t = \Delta_{\mathbb{G}} u_t$ with initial datum $u_0 \in L^1(\mathbb{G}, \mathscr{L}^n)$;
- μ_t a GF of Ent in $(\mathscr{P}_2(\mathbb{G}), \mathsf{W}_2)$ starting from $\mu_0 = u_0 \mathscr{L}^n \in \mathsf{Dom}(\mathsf{Ent})$.

Then the two GFs are unique and coincide, i.e. $\mu_t = u_t \mathscr{L}^n$ for all $t > 0$.

Idea of the proof: GF heat $\Rightarrow$ GF Ent

Assume $(u_t)_{t>0}$ solves $\partial_t u_t = \Delta_{\mathbb{G}} u_t$ with $u_0 \in L^1(\mathbb{G})$, $u_0 \mathscr{L}^n \in \mathrm{Dom}(\mathrm{Ent})$
$\Rightarrow u_t(x) = u_0 \star \mathsf{h}_t(x) = \int_{\mathbb{G}} \mathsf{h}_t(y^{-1}x) u_0(y)\, dy$, where $(\partial_t - \Delta_{\mathbb{G}})\mathsf{h}_t = \delta_0$
$\Rightarrow$ by Gaussian estimates on h_t (Varopoulos, Saloff-Coste, Coulhon)

$$\frac{d}{dt}\mathrm{Ent}(\mu_t) \overset{\text{(HE)}}{=} \int_{\mathbb{G}} (1 + \log u_t)\, \Delta_{\mathbb{G}} u_t \, dx = -\int_{\mathbb{G}} \frac{\|\nabla_{\mathbb{G}} u_t\|_{\mathbb{G}}^2}{u_t}\, dx.$$

Since $\partial_t u_t + \mathrm{div}(v_t u_t) = 0$ with $v_t = -\frac{\nabla_{\mathbb{G}} u_t}{u_t}$, the velocity of $(\mu_t)_{t>0}$ satisfies

$$|\dot{\mu}_t|^2 \le \int_{\mathbb{G}} \|v_t\|_{\mathbb{G}}^2 \, d\mu_t = \int_{\mathbb{G}} \frac{\|\nabla_{\mathbb{G}} u_t\|_{\mathbb{G}}^2}{u_t}\, dx \quad \text{for a.e. } t > 0.$$

By the (metric) chain rule

$$\left| \frac{d}{dt}\mathrm{Ent}(\mu_t) \right| \le |\mathsf{D}_{\mathbb{G}}^- \mathrm{Ent}|(\mu_t) \cdot |\dot{\mu}_t| \qquad \text{for a.e. } t > 0.$$

We conclude by proving that

$$|\mathsf{D}_{\mathbb{G}}^- \mathrm{Ent}|^2(\mu_t) = \int_{\mathbb{G}} \frac{\|\nabla_{\mathbb{G}} u_t\|_{\mathbb{G}}^2}{u_t}\, dx \qquad \leftarrow \text{Fisher information!}$$

($\ge$ is always true, $\le$ needs Riemannian approximation of $\mathbb{G}$ and estimates on h_t)

Idea of the proof: GF Ent $\Rightarrow$ GF heat

Assume $(\mu_t)_{t>0}$ is a GF of Ent with $u_0 \in L^1(\mathbb{G})$, $u_0 \mathscr{L}^n \in \mathrm{Dom}(\mathrm{Ent})$

$\Rightarrow \mu_t = u_t \mathscr{L}^n$ since $\mathrm{Ent}(\mu) < +\infty \iff \mu \ll \mathscr{L}^n$ by definition

$\Rightarrow (u_t)_{t>0}$ satisfies (CE) for some v_t and for a.e. $t > 0$

$$-\frac{d}{dt}\mathrm{Ent}(\mu_t) \geq \frac{1}{2}|\dot{\mu}_t|^2 + \frac{1}{2}|\mathsf{D}^-\mathrm{Ent}(\mu_t)|^2 \geq \|v_t\|_{L^2(\mu_t)} \cdot \left\| \frac{\nabla_{\mathbb{G}} u_t}{u_t} \right\|_{L^2(\mu_t)}$$

Smooth in time and space (u_t, v_t) and get $(u_t^\varepsilon, v_t^\varepsilon)_{\varepsilon>0}$ (using group structure!)

$\Rightarrow (\mu_t^\varepsilon)_{t>0}$ is not a GF, but still $\partial_t u_t^\varepsilon + \mathrm{div}(v_t^\varepsilon u_t^\varepsilon) = 0$, hence

$$\int_{\mathbb{R}} \int_{\mathbb{G}} \langle \nabla_{\mathbb{G}} \phi_t, v_t^\varepsilon \rangle_{\mathbb{G}} u_t^\varepsilon \; dx dt \overset{\text{(CE)}}{=} -\int_{\mathbb{R}} \int_{\mathbb{G}} \partial_t \phi_t \, u_t^\varepsilon \; dx dt = \int_{\mathbb{R}} \int_{\mathbb{G}} \phi_t \, \partial_t u_t^\varepsilon \; dx dt$$

$\Rightarrow$ by a cut-off argument, we test $\phi_t = (1 + \log u_t^\varepsilon)$ and get

$$\frac{d}{dt}\mathrm{Ent}(\mu_t^\varepsilon) = \int_{\mathbb{G}} (1 + \log u_t^\varepsilon) \, \partial_t u_t^\varepsilon \; dx = \int_{\mathbb{G}} \left\langle \frac{\nabla_{\mathbb{G}} u_t^\varepsilon}{u_t^\varepsilon}, v_t^\varepsilon \right\rangle d\mu_t^\varepsilon$$

$\Rightarrow$ we pass to the limit as $\varepsilon \to 0$ and conclude by Cauchy-Schwarz

Related problems and future developements

- Can we find a "generalised CD condition" for sub-Riemmanian manifolds?
  - ★ Transverse symmetries: [Baudoin-Garofalo, 2014]
    → Generalised BE inequalities
  - ★ Special structures: [Balogh-Kristaly-Sipos] & [Barilari-Rizzi]
    → Modified interpolation inequalities
- Does the flow correspondence hold in other sub-Riemannian manifolds?
- Can we use the generalised CD-type conditions above to prove it?

Thank you for your attention!