Space regularity for evolution operators of Hörmander type with coefficients measurable in time

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Introduction

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- The same procedure works if the coefficients a_{ij} are not constant but only depend on t: under the assumption

$$\nu |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(t) \xi_i \xi_j \leq \nu^{-1} |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, t \in (0, T), \text{ some } \nu > 0$$

the heat kernel of H can be explicitly computed, with no regularity assumption on $a_{ij} \in L^\infty(0, T)$.

Example

The solution to

$$\begin{cases} \partial_t u - a(t)\,\Delta u = 0 \text{ in } (0,T)\times\mathbb{R}^n \\ u(0,x) = f(x) \in L^1(\mathbb{R}^n) \end{cases} \quad \text{with } 0<\nu \le a(t) \le \nu^{-1}$$

is given by

$$u(t,x) = \frac{1}{(4\pi A(t))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4A(t)}} f(y)\,dy \text{ with } A(t) = \int_0^t a(\tau)\,d\tau.$$

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- Space regularity still holds if $Hu(t,x) = F(t,x)$ with F smooth in x and, e.g., $L^2((0,T)\times\mathbb{R}^n)$.
- We want to study the analogous properties for degenerate parabolic operators structured on Hörmander vector fields on Carnot groups

$$Hu = \partial_t u - \sum_{i,j=1}^{q} a_{ij}(t)\,X_i X_j u.$$

Preliminaries - Hörmander operators

Example

Let us consider the degenerate elliptic operator in $\mathbb{R}^3$

$$L = (\partial_x + 2y\partial_t)^2 + (\partial_y - 2x\partial_t)^2$$

which we rewrite as sum of squares of two vector fields:

$$L = X_1^2 + X_2^2$$

Observe that

$$[X_1, X_2] = X_1X_2 - X_2X_1 = -4\partial_t.$$

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- This fact is responsible for the good regularization properties of L.

Hörmander's theorem (Acta Math. 1967)

- Let $X_0, X_1, \ldots, X_q$ be real vector fields with C^∞ coefficients in a domain $\Omega \subseteq \mathbb{R}^n$. Let

$$L = \sum_{j=1}^{q} X_j^2 + X_0$$

and assume that at any point of Ω, among the iterated commutators X_{j_1}, $[X_{j_1}, X_{j_2}]$, $[X_{j_1}, [X_{j_2}, X_{j_3}]]$, ... there exist n which are linearly independent ("Hörmander's condition").

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- Then L is *hypoelliptic* in Ω, that is: for every distribution u such that Lu is C^∞ in an open $A \subseteq \Omega$ then also u is $C^\infty(A)$.

Carnot groups

- A homogeneous group (in $\mathbb{R}^N$) is a Lie group $\left(\mathbb{R}^N, \circ\right)$ (where $\circ$ is thought as "translation") endowed with a family $\{D_\lambda\}_{\lambda>0}$ of group automorphisms ("dilations") given by:

$$D_\lambda \left(x_1, x_2, ..., x_N\right) = \left(\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, ..., \lambda^{\alpha_N} x_N\right) \tag{1}$$

for integers $1 = \alpha_1 \leqslant \alpha_2 \leqslant ... \leqslant \alpha_N$.

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  for integers $1 = \alpha_1 \leqslant \alpha_2 \leqslant ... \leqslant \alpha_N$.
- We will denote by $\mathbb{G} = \left(\mathbb{R}^N, \circ, D_\lambda\right)$ this structure. The number

  $Q = \sum_{i=1}^N \alpha_i$ is called homogeneous dimension of $\mathbb{G}$.

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Example

Heisenberg group $\mathbb{H}^n$: in $\mathbb{R}^{n+n+1} \ni (x, y, t)$ set

$$\begin{aligned} (x, y, t) \circ (x', y', t') &= (x + x', y + y', t + t' - 2(x \cdot y' - x' \cdot y)) \\ D_\lambda(x, y, t) &= (\lambda x, \lambda y, \lambda^2 t) \\ N &= 2n + 1; Q = 2n + 2 \end{aligned}$$

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- A differential operator P on $\mathbb{R}^N$ is said left invariant if for every smooth function f

$$P\left(L_y f\right)(x) = L_y\left(Pf\left(x\right)\right)\ \forall x, y \in \mathbb{R}^N,$$

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$$L_y f\left(x\right) = f\left(y \circ x\right).$$

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- Let us denote by X_i ($i = 1, 2, ..., N$) the only left invariant vector field which agrees at 0 with ∂_{x_i}.
- Analogoulsy X_i^R is the only right invariant vector field which agrees at 0 with ∂_{x_i}, and then with X_i.

Carnot groups

Example (Heisenberg group $\mathbb{H}^1$)

In $\mathbb{R}^3 \ni (x, y, t)$ set

$$\begin{aligned}(x, y, t) \circ (x', y', t') &= (x + x', y + y', t + t' - 2(xy' - x'y)) \\ D_\lambda (x, y, t) &= (\lambda x, \lambda y, \lambda^2 t)\end{aligned}$$

then

$$\begin{array}{ll} X_1 = \partial_x + 2y\partial_t & X_1^R = \partial_x - 2y\partial_t \\ X_2 = \partial_y - 2x\partial_t & X_2^R = \partial_y + 2x\partial_t \\ X_3 = \partial_t & X_3^R = \partial_t \end{array}$$

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- Then we say that $\mathbb{G}$ is a Carnot group of step s, $X_1, ..., X_q$ its generators and

$$L = \sum_{i=1}^{q} X_i^2$$

is the canonical sublaplacian on $\mathbb{G}$. It is *hypoelliptic*, *left invariant*, 2-*homogeneous*.

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Example (The Konh Laplacian on the Heisenberg group)

$$L = \sum_{i=1}^{n} \left(X_i^2 + Y_i^2 \right) \text{ in } \mathbb{R}^{2n+1}$$

with $X_i = \partial_{x_i} + 2y_i\partial_t$; $Y_i = \partial_{y_i} - 2x_i\partial_t$; $[X_i, Y_i] = -4\partial_t$;

$q = 2n < N = 2n + 1; s = 2.$

Carnot groups

- Analogously, the *heat-type operator*

$$H = \sum_{i=1}^{q} X_i^2 - \partial_t$$

is *hypoelliptic*, *left invariant*, 2-*homogeneous on* $\mathbb{R} \times \mathbb{G}$ endowed with the structure of homogeneous group:

$$\begin{aligned} (t, x) * (s, y) &= (t + s, x \circ y) \\ D_\lambda^* (t, x) &= \left(t^2, D_\lambda (x)\right). \end{aligned}$$

Objective of the research

- Let $X_1, ..., X_q$ be the generators of a a Carnot group $\mathbb{G}$ in $\mathbb{R}^N$ $(N > q)$ and let

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- We want to prove a regularity result for $\mathcal{L}$ in the space variables, with quantitative regularity estimates on u in terms of $\mathcal{L}u$.

Comparison with the existing literature

- Krylov (SIAM J. Math. Anal. 2014) has proved a Hörmander theorem (w.r.t. space variables) for operators

$$\mathcal{L} = \partial_t - \sum_{k=1}^{q} L_k^2 + L_0$$

with

$$L_k = \sum_{i=1}^{N} \sigma^{ik}(t, x)\, \partial_{x_i}$$

where $\sigma^{ik}(t, x)$ are assumed to have x-derivatives of every order uniformly bounded for $(t, x) \in (0, 1) \times \mathbb{R}^N$, and the vector fields $L_0, L_1, \ldots, L_q$ satisfy Hörmander's condition in $\mathbb{R}^N$ for every fixed t.

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- Krylov exploits techniques of pseudodifferential operators, like in Kohn's proof of Hörmander's theorem.
- The generators of a Carnot group do not have bounded x-derivatives.
- I am interested in adapting to this situation a technique of proof, well suited to Carnot groups, used in [Bramanti-Brandolini, Nonlin. Anal. 2015].

Some motivation to study evolution equations with irregular time dependent coefficients

- Krylov has applied his regularity result to prove an analogous result for stochastic PDEs [Algebra i Analiz 2015], which in turn is applied to filtering theory of partially observable diffusion processes [Prob. Theory Related Fields 2015], in a context where there is no good control on the continuity modulus of coefficients w.r.t. time.

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- Hyperbolic operators of the kind

$$Hu = u_{tt} - \sum_{i,j=1}^{n} a_{ij}\left(t\right) u_{x_i x_j}$$

with merely bounded measurable a_{ij} have been studied by many authors, see for instance [Colombini, De Giorgi, Spagnolo, Ann. Pisa 1979], [Garetto, Ruzhansky, Arch. Ration. Mech. Anal., 2015] where motivation from geophysics and tomography are given for this study.

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- Again Krylov has used parabolic operators with coefficients $a_{ij}(t) \in L^{\infty}$ as model operators to study nonvariational parabolic operators with coefficients $a_{ij}(t,x)$ VMO w.r.t. x and L^{∞} in t.

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- The operators studied in the present paper can also be seen as model operators to study the more general class

$$\mathcal{L} = \sum_{i,j=1}^{q} a_{ij}(t,x) X_i X_j - \partial_t,$$

under weaker conditions with respect to time.

The starting point

Lemma

Any two differential operators $\mathcal{L}, \mathcal{R}$ left and right invariant, respectively, commute:

$$\mathcal{L}\mathcal{R}f = \mathcal{R}\mathcal{L}f.$$

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Lemma

Let $u \in L^2\left((0,T), W_X^{2,2}(\mathbb{G})\right) \cap W^{1,2}\left((0,T), L^2(\mathbb{G})\right)$*,* $u(0,\cdot) = 0$*, then*

$$\left\|\nabla_X u\right\|_{L^2(\mathbb{G}_T)} \equiv \sum_{i=1}^{q} \left\|X_i u\right\|_{L^2(\mathbb{G}_T)} \leqslant c_\nu \left\{\left\|\mathcal{L}u\right\|_{L^2(\mathbb{G}_T)} + \left\|u\right\|_{L^2(\mathbb{G}_T)}\right\}$$

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- Assume for the moment we knew the (apparently similar) estimate

$$\|\nabla_{X^R} u\|_{L^2(\mathbb{G}_T)} \equiv \sum_{i=1}^{q} \left\|X_i^R u\right\|_{L^2(\mathbb{G}_T)} \leqslant c_\nu \left\{ \|\mathcal{L}u\|_{L^2(\mathbb{G}_T)} + \|u\|_{L^2(\mathbb{G}_T)} \right\}$$

- We could then apply this estimate to $X_{i_1}^R X_{i_2}^R ... X_{i_k}^R u$, getting

$$\left\| X_i^R X_{i_1}^R ... X_{i_k}^R u \right\|_{L^2(\mathbb{G}_T)} \leqslant c \left\{ \left\| \mathcal{L} X_{i_1}^R ... X_{i_k}^R u \right\|_{L^2(\mathbb{G}_T)} + \left\| X_{i_1}^R ... X_{i_k}^R u \right\|_{L^2(\mathbb{G}_T)} \right\}$$

and since left and write invariant operators, and also ∂_t and $X_{i_1}^R$, commute

$$= c \left\{ \left\| X_{i_1}^R ... X_{i_k}^R \mathcal{L} u \right\|_{L^2(\mathbb{G}_T)} + \left\| X_{i_1}^R ... X_{i_k}^R u \right\|_{L^2(\mathbb{G}_T)} \right\}$$

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- so that

$$\|u\|_{L^2\left((0,T),W_{X^R}^{k+1,2}(\mathbb{G})\right)} \leqslant c \left(\|\mathcal{L}u\|_{L^2\left((0,T),W_{X^R}^{k,2}(\mathbb{G})\right)} + \|u\|_{L^2\left((0,T),W_{X^R}^{k,2}(\mathbb{G})\right)} \right.$$

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- We could then apply this estimate to $X_{i_1}^R X_{i_2}^R \dots X_{i_k}^R u$, getting

$$\left\| X_i^R X_{i_1}^R \dots X_{i_k}^R u \right\|_{L^2(\mathbb{G}_T)} \leqslant c \left\{ \left\| \mathcal{L} X_{i_1}^R \dots X_{i_k}^R u \right\|_{L^2(\mathbb{G}_T)} + \left\| X_{i_1}^R \dots X_{i_k}^R u \right\|_{L^2(\mathbb{G}_T)} \right\}$$

and since left and write invariant operators, and also ∂_t and $X_{i_1}^R$, commute

$$= c \left\{ \left\| X_{i_1}^R \dots X_{i_k}^R \mathcal{L} u \right\|_{L^2(\mathbb{G}_T)} + \left\| X_{i_1}^R \dots X_{i_k}^R u \right\|_{L^2(\mathbb{G}_T)} \right\}$$

- so that

$$\left\| u \right\|_{L^2\left((0,T),W^{k+1,2}_{X^R}(\mathbb{G})\right)} \leqslant c \left(\left\| \mathcal{L} u \right\|_{L^2\left((0,T),W^{k,2}_{X^R}(\mathbb{G})\right)} + \left\| u \right\|_{L^2\left((0,T),W^{k,2}_{X^R}(\mathbb{G})\right)} \right)$$

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- This is the key idea of this technique: measuring the degree of regularity of u in terms of $\mathcal{L}u$ (with $\mathcal{L}$ left invariant) using right invariant derivatives apparently trivializes the problem; actually it doesn't, but greatly simplifies the proof of higher order estimates.

Regularity estimates

- The problem is that actually do not have the estimate

$$\sum_{i=1}^{q} \left\| X_i^R u \right\|_{L^2(\mathbb{G}_T)} \leqslant c_\nu \left\{ \| \mathcal{L} u \|_{L^2(\mathbb{G}_T)} + \| u \|_{L^2(\mathbb{G}_T)} \right\}$$

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- We will see that nevertheless it is possible to control the regularity using right invariant vector fields, but this requires asking a much higher regularity to $\mathcal{L}u$. Namely, we can prove the following:

Main result

Theorem

Let $u \in W^{1,2}\left(\left(0,T\right), L^2_{loc}\left(\mathbb{G}\right)\right)$, $u\left(0,\cdot\right)=0$, *be a weak solution to* $\mathcal{L}u = F \in L^2\left(\left(0,T\right), L^2_{loc}\left(\mathbb{G}\right)\right)$. *Then*

1. $\forall k = 1,2,3,...$*and* $\zeta, \zeta_1 \in C_0^{\infty}\left(\mathbb{G}\right), \zeta \prec \zeta_1$,

$$\zeta_1 F \in L^2\left(\left(0,T\right), W^{k+s^2-1,2}_{X^R}\left(\mathbb{G}\right)\right) \Longrightarrow \zeta u \in L^2\left(\left(0,T\right), W^{k,2}_{X^R}\left(\mathbb{G}\right)\right)$$

and

$$\left\|\zeta u\right\|_{L^2\left(\left(0,T\right), W^{k,2}_{X^R}\left(\mathbb{G}\right)\right)} \leqslant c\left\{\left\|\zeta_1 F\right\|_{L^2\left(\left(0,T\right), W^{k+s-1,2}_{X^R}\left(\mathbb{G}\right)\right)} + \left\|\zeta_1 u\right\|_{L^2\left(\mathbb{G}_T\right)}\right\}$$

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2. *In particular if* $k \geq 2s$ *then* u *is also a strong solution to the equation;*

Theorem

3 $\forall \zeta, \zeta_1 \in C_0^\infty(\mathbb{G}), \zeta \prec \zeta_1$ *and multiindex* α *there exists* $c > 0$ *and an integer* $h > 0$ *such that if* $F \in L^2\left((0, T), W^{h,2}_{X^R,loc}(\mathbb{G})\right)$ *then*

$$\sup_{0<t_1<t_2<T} \sup_{x\in\mathbb{G}} \frac{\left|\zeta(x)\left[\partial_x^\alpha u(t_2, x) - \partial_x^\alpha u(t_1, x)\right]\right|}{\left|t_2 - t_1\right|^{1/2}}$$
$$\leqslant c\left\{\left\|\zeta_1 F\right\|_{L^2\left((0,T),W^{h,2}_{X^R}(\mathbb{G})\right)} + \left\|\zeta_1 u\right\|_{L^2(\mathbb{G}_T)}\right\}$$

and for every $t \in [0, T]$

$$\sup_{x\in\mathbb{G}} \left|\zeta(x)\,\partial_x^\alpha u(t, x)\right| \leqslant c\,|t|^{1/2}\left\{\left\|\zeta_1 F\right\|_{L^2\left((0,T),W^{h,2}_{X^R}(\mathbb{G})\right)} + \left\|\zeta_1 u\right\|_{L^2(\mathbb{G}_T)}\right\}.$$

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4.

$$\textit{If } \zeta_1 F\in L^2\left((0,T),C^\infty(\mathbb{G})\right),\ \textit{then}$$
$$\zeta u\in C^0\left([0,T],C^\infty(\mathbb{G})\right)\ \textit{and}\ \zeta u_t\in L^2\left((0,T),C^\infty(\mathbb{G})\right).$$

The regularity result is optimal

Example

Let us consider the uniformly parabolic operator

$$\mathcal{L}u = -u_t + a(t)\, u_{xx}$$

with $a \in L^\infty(\mathbb{R})$, $0 < \nu \leq a(t) \leq \nu^{-1}$. The function

$$u(t,x) = \exp\left(-\int_0^t a(\tau)\, d\tau\right) \sin x$$

satisfies $\mathcal{L}u = 0$; u is smooth in x and only Lipschitz continuous in t.

Example

Let now

$$U(t,x) = t^{\alpha} u(t,x) \text{ for some } \alpha \in \left(\frac{1}{2}, 1\right).$$

Then U solves the problem

$$\begin{cases} \mathcal{L}U = F \text{ for } x \in \mathbb{R}, t > 0 \\ U(0,x) = 0 \end{cases}$$

with $F(t,x) = -\alpha t^{\alpha-1} u(t,x)$.
We have

$$\begin{aligned} F &\in L^2((0,1) \times \mathbb{R}) \ \forall \alpha > \frac{1}{2} \text{ and} \\ U &\in W^{1,2}((0,T), C^{\infty}(\mathbb{R})) \cap C^{0,\alpha}([0,T], C^{\infty}(\mathbb{R})). \end{aligned}$$

Space regularity - Idea of the proof

- We have adapted to the parabolic case the technique used in [B.-Brandolini, Nonlin. Anal. 2015].

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- For a fixed increment $h \in \mathbb{G}$ let

$$\Delta_h f(t,x) = f(t, x \circ h) - f(t,x)$$
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- If $h = \mathrm{Exp}\left(tX_i\right)$ then one can easily prove

$$\left\|\Delta_h f\right\|_{L^2(\mathbb{G}_T)} \leqslant |t| \left\|X_i f\right\|_{L^2(\mathbb{G}_T)} \leqslant \|h\| \left\|X_i f\right\|_{L^2(\mathbb{G}_T)}$$

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- where $\|h\|$ is a homogeneous norm on $\mathbb{G}$, and

$$\|X_i f\|_{L^2(\mathbb{G}_T)} \leqslant \sup_{\substack{h=\operatorname{Exp}(tX_i) \\ 0<|t|<1}} \frac{\|\Delta_h f\|_{L^2(\mathbb{G}_T)}}{\|h\|}; \text{(analogously for } X_i^R \text{ / } \widetilde{\Delta}_h\text{)}$$

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- A technical lemma then allows to get from this the following

$$\left\|\widetilde{\Delta}_h u\right\|_{L^2(\mathbb{G}_T)} \leqslant c \, \|h\|^{1/s} \left(\|\mathcal{L}u\|_{L^2(\mathbb{G}_T)} + \|u\|_{L^2(\mathbb{G}_T)}\right)$$

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for u compactly supported, c depending on $\operatorname{sprt} u$.
- To get a control on $\left\|X_i^R u\right\|_{L^2(\mathbb{G}_T)}$ we need $\|h\|$ instead of $\|h\|^{1/s}$.
- Then, for fixed $h = \operatorname{Exp}(tX_i)$, $i = 1, \dots, q$, consider the iterated finite difference $\widetilde{\Delta}_h^m u = \widetilde{\Delta}_h \widetilde{\Delta}_h^{m-1} u$.

Space regularity - Idea of the proof

- From the previous inequality we get

$$\left\| \widetilde{\Delta}_h^m u \right\|_{L^2(\mathbb{G}_T)} \leqslant c \left\| h \right\|^{1/s} \left(\left\| \mathcal{L} \widetilde{\Delta}_h^{m-1} u \right\|_{L^2(\mathbb{G}_T)} + \left\| \widetilde{\Delta}_h^{m-1} u \right\|_{L^2(\mathbb{G}_T)} \right)$$

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Space regularity - Idea of the proof

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- In particular $(m = s+1)$

$$\left\|\widetilde{\Delta}_h^{s+1} u\right\|_{L^2(\mathbb{G}_T)} \leqslant c\,\|h\|^{1+1/s}\left(\|\mathcal{L}u\|_{L^2\left((0,T),W_{XR}^{s,2}(\mathbb{G})\right)} + \|u\|_{L^2(\mathbb{G}_T)}\right).$$

Space regularity - Idea of the proof

- From the previous inequality we get

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- **Lemma**. If $1 < \alpha < 2$ and

$$\begin{aligned} \left\| \widetilde{\Delta}_h^m u \right\|_{L^2(\mathbb{G}_T)} &\leqslant A \left\| h \right\|^{\alpha}, \text{ then} \\ \left\| \widetilde{\Delta}_h u \right\|_{L^2(\mathbb{G}_T)} &\leqslant c \left(A + \left\| u \right\|_{L^2(\mathbb{G}_T)} \right) \left\| h \right\|. \end{aligned}$$

Space regularity - Idea of the proof

- Hence we get

$$\left\|\widetilde{\Delta}_h u\right\|_{L^2(\mathbb{G}_T)} \leqslant c\,\|h\|\left(\|\mathcal{L}u\|_{L^2\left((0,T),W^{s,2}_{X^R}(\mathbb{G})\right)} + \|u\|_{L^2(\mathbb{G}_T)}\right)$$

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$$\|\zeta_1 u\|_{L^2\left((0,T),W^{1,2}_{X^R}(\mathbb{G})\right)} \leqslant c \left\{ \|\zeta_2 \mathcal{L}u\|_{L^2\left((0,T),W^{s,2}_{X^R}(\mathbb{G})\right)} + \|\zeta_2 u\|_{L^2(\mathbb{G}_T)} \right\}$$

Space regularity - Idea of the proof

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- and this is the first step to prove, iteratively, the desired estimate.
- Iteration is now easy because the vector fields X^R commute with $\mathcal{L}$.

Space regularity - Idea of the proof

- This gives the first part of the result:

Theorem

Let $u \in W^{1,2}\left(\left(0,T\right), L^2_{loc}\left(\mathbb{G}\right)\right)$, $u\left(0,\cdot\right)=0$, *be a weak solution to* $\mathcal{L}u = F \in L^2\left(\left(0,T\right), L^2_{loc}\left(\mathbb{G}\right)\right)$. *Then* $\forall k = 1,2,3,\ldots$*and* $\zeta, \zeta_1 \in C_0^\infty\left(\mathbb{G}\right), \zeta \prec \zeta_1$,

$$\zeta_1 F \in L^2\left(\left(0,T\right), W_{X^R}^{k+s^2-1,2}\left(\mathbb{G}\right)\right) \Longrightarrow \zeta u \in L^2\left(\left(0,T\right), W_{X^R}^{k,2}\left(\mathbb{G}\right)\right) \text{ and}$$

$$\left\|\zeta u\right\|_{L^2\left((0,T),W_{X^R}^{k,2}(\mathbb{G})\right)} \leqslant c\left\{\left\|\zeta_1 F\right\|_{L^2\left((0,T),W_{X^R}^{k+s-1,2}(\mathbb{G})\right)} + \left\|\zeta_1 u\right\|_{L^2(\mathbb{G}_T)}\right\}.$$

Hölder continuity in time - Idea of the proof

- Once we know the space regularity of u, to get Hölder estimates with respect to t it is enough to integrate in t the equation:

$$\begin{aligned} u(t_2,x)-u(t_1,x) &= \int_{t_1}^{t_2} \partial_t u(t,x)\,dt \\ X_I^R u(t_2,x)-X_I^R u(t_1,x) &= \int_{t_1}^{t_2} \partial_t X_I^R u(t,x)\,dt. \end{aligned}$$

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-

$$\begin{aligned} &\int_{\mathbb{G}}\zeta(x)^2\left|X_I^R u(t_2,x)-X_I^R u(t_2,x)\right|^2 dx \\ &\leqslant \int_{\mathbb{G}}\zeta(x)^2\left|\int_{t_1}^{t_2}\left\{-X_I^R\mathcal{L}u(t,x)+\sum_{i,j=1}^{q}a_{ij}(t)\,X_iX_jX_I^R u(t,x)\right\}dt\right|^2 dx \end{aligned}$$

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- $$\int_{\mathbb{G}} \zeta(x)^2 \left| X_I^R u(t_2,x) - X_I^R u(t_2,x) \right|^2 dx$$
$$\leqslant \int_{\mathbb{G}} \zeta(x)^2 \left| \int_{t_1}^{t_2} \left\{ -X_I^R \mathcal{L} u(t,x) + \sum_{i,j=1}^{q} a_{ij}(t) X_i X_j X_I^R u(t,x) \right\} dt \right|^2 dx$$

- $$\leqslant |t_2-t_1| \left\{ \left\| \zeta X_I^R F \right\|_{L^2(\mathbb{G}_T)}^2 + c_\nu \sum_{i,j=1}^{q} \left\| \zeta X_i X_j X_I^R u \right\|_{L^2(\mathbb{G}_T)}^2 \right\}.$$

- By the space regularity estimates, this implies that

$$\sup_{0<t_1<t_2<T} \frac{\int_{\mathbb{G}} \zeta(x)^2 \left| X_I^R u(t_2,x) - X_I^R u(t_2,x) \right|^2 dx}{|t_2 - t_1|}$$

$$\leqslant c \left\{ \|\zeta_1 F\|_{L^2\left((0,T),W^{h,2}_{X^R}(\mathbb{G})\right)} + \|\zeta_1 u\|_{L^2(\mathbb{G}_T)} \right\}^2$$

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- Since every derivative $\partial_x^\alpha v(x)$ can be bounded, uniformly on compact sets, by a suitable linear combination of X_I^R, we get

$$\sup_{0<t_1<t_2<T} \frac{\left\|\zeta\left[\partial_x^\alpha u(t_2,\cdot) - \partial_x^\alpha u(t_1,\cdot)\right]\right\|_{L^2(\mathbb{G})}}{|t_2-t_1|^{1/2}}$$

$$\leqslant c \left\{ \|\zeta_1 F\|_{L^2\left((0,T),W^{h_1,2}_{X^R}(\mathbb{G})\right)} + \|\zeta_1 u\|_{L^2(\mathbb{G}_T)} \right\}$$

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for some integer $h_1 > h$.

- By Sobolev embeddings, this also gives a control on

$$\sup_{0<t_1<t_2<T} \sup_{x\in\mathbb{G}} \frac{\left|\zeta(x)\left[\partial_x^\alpha u(t_2,x) - \partial_x^\alpha u(t_1,x)\right]\right|}{|t_2 - t_1|^{1/2}}.$$

Extension to distributional solutions

- Our smoothness result, proved for functions in $W^{1,2}\left((0,T), L^2_{loc}(\mathbb{G})\right)$, can be extended to more general distributions.

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- Our smoothness result, proved for functions in $W^{1,2}\left((0,T), L^2_{loc}(\mathbb{G})\right)$, can be extended to more general distributions.

Definition

Let $\Omega \subseteq \mathbb{G}$ be an open set. We say that $u \in L^2\left((0,T), \mathcal{D}'(\Omega)\right)$ if $u \in \mathcal{D}'(\Omega_T)$ and $\forall \phi \in \mathcal{D}(\Omega)\ \exists\ h_\phi \in L^2(0,T)$ s.t.

$$\langle u, \phi \otimes \psi \rangle = \int_0^T h_\phi(t)\, \psi(t)\, dt\ \forall \psi \in \mathcal{D}(0,T).$$

We will write $h_\phi(t) = \langle u(t,\cdot), \phi \rangle$ and

$$\langle u, \phi(x)\, \psi(t) \rangle = \int_0^T \langle u(t,\cdot), \phi \rangle\, \psi(t)\, dt\ \forall \phi \in \mathcal{D}(\Omega), \psi \in \mathcal{D}(0,T),$$

Analogously, we say that $u \in W^{1,2}\left((0,T), \mathcal{D}'(\Omega)\right)$ if $u \in \mathcal{D}'(\Omega_T)$ with $u, \partial_t u \in L^2\left((0,T), \mathcal{D}'(\Omega)\right)$.

Extension to distributional solutions

Definition

We say that u is a distributional solution to $\mathcal{L}u = F$ in $\Omega_T = (0, T) \times \Omega$, with $F \in L^2\left((0, T), \mathcal{D}'(\Omega)\right)$ if $u \in W^{1,2}\left((0, T), \mathcal{D}'(\Omega)\right)$ and:

$$\langle -\partial_t u(t, \cdot), \phi \rangle + \sum_{i,j=1}^{q} a_{ij}(t) \langle X_i X_j u(t, \cdot), \phi \rangle = \langle F(t, \cdot), \phi \rangle$$

$\forall \phi \in \mathcal{D}(\Omega)$ and a.e. $t \in (0, T)$.

Extension to distributional solutions

Definition

We say that $u \in L^2((0,T), \mathcal{D}'(\Omega))$ satisfies the x-finite order assumption on Ω if $\exists h \in L^2\left((0,T), L^1_{loc}(\Omega)\right)$ and a multiindex α s.t.

$$u = \frac{\partial^\alpha h}{\partial x^\alpha} \text{ in } \mathcal{D}'(\Omega_T).$$

If $u \in W^{1,2}((0,T), \mathcal{D}'(\Omega))$, we say that u satisfies the x-finite order assumption on Ω if the above condition holds with $h \in W^{1,2}\left((0,T), L^1_{loc}(\Omega)\right)$.

Extension to distributional solutions

Theorem

For some bounded domain $\Omega \subset \mathbb{G}$, let u be a distributional solution to $\mathcal{L}u = F$ in Ω_T with $F \in L^2\left((0,T), \mathcal{D}'(\Omega)\right)$. Assume that u satisfies the x-finite order assumption and $u(0,\cdot) = 0$ in Ω. Then, for every domain $\Omega' \Subset \Omega$, if

$$F \in L^2\left((0,T), C^\infty\left(\overline{\Omega}\right)\right)$$

then

$$u \in C^0\left([0,T], C^\infty\left(\overline{\Omega'}\right)\right) \text{ and } u_t \in L^2\left((0,T), C^\infty\left(\overline{\Omega'}\right)\right).$$