Some properties of optimal expansions in non-integer bases

Based on joint work with K. Dajani, V. Komornik and P. Loreti

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Outline

1. Summary

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ABSTRACT

Given a base $q \in (1, 2]$ and a real number $x \in J_q := [0, 1/(q-1)]$, we call a sequence $(c_i) = c_1 c_2 \ldots$ of zeros and ones an *optimal expansion* of x if

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Our main result states that, except for a countable set of bases in which every number in J_q has an optimal expansion, "most" numbers have no optimal expansion.

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- By an *expansion of a real number x (in base q with respect to A)* we mean a sequence (c_i) satisfying

$$\frac{c_1}{q} + \frac{c_2}{q^2} + \frac{c_3}{q^3} + \cdots = x.$$

The numbers c_i are sometimes called the *digits* of the expansion (c_i).

EXISTENCE OF EXPANSIONS

Proposition

Each number $x \in J_{A,q} := [a_0/(q-1), a_m/(q-1)]$ *has at least one expansion if and only if the Pedicini condition holds:*

$$\max_{1 \le j \le m} (a_j - a_{j-1}) \le \frac{a_m - a_0}{q-1}.$$

PROOF OF NECESSITY

The condition

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if $a_\ell - a_{\ell-1} > (a_m - a_0)/(q-1)$ for some ℓ, then none of the numbers in the nonempty interval

$$\left(\frac{a_{\ell-1}}{q} + \sum_{i=2}^{\infty}\frac{a_m}{q^i}, \frac{a_\ell}{q} + \sum_{i=2}^{\infty}\frac{a_0}{q^i}\right)$$

has an expansion.

PROOF OF SUFFICIENCY

- Suppose that $x \in J_{A,q}$. We define recursively an expansion (b_i) of x by applying the *greedy algorithm* of Rényi:
  if for some positive integer n, b_i is already defined for $i < n$, then b_n is the largest digit in A satisfying $\sum_{i=1}^{n} b_i q^{-i} \leq x$.

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  - replacing a_i by $a_i - a_0$ for each i does not alter the inequality $\max_{1 \leq j \leq m} \left(a_j - a_{j-1} \right) \leq \frac{a_m - a_0}{q-1}$.

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  - x has an expansion with respect to alphabet A if and only if $x - a_0/(q-1)$ has an expansion with respect to the alphabet $\{0, a_1 - a_0, a_2 - a_0, \ldots, a_m - a_0\}$.

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  - If $b_n < a_m$ for infinitely many n, we see that (b_i) is an expansion of x by letting $n \to \infty$ along these indices.

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- If there were a last index n such that $b_n = a_j < a_m$, then

$$\left(\sum_{i=1}^{n} \frac{b_i}{q^i}\right) + \sum_{i=n+1}^{\infty} \frac{a_m}{q^i} \leq x < \left(\sum_{i=1}^{n} \frac{b_i}{q^i}\right) + \frac{a_{j+1} - a_j}{q^n}$$

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which violates the condition $\max_{1 \leq j \leq m} \left(a_j - a_{j-1}\right) \leq \frac{a_m}{q-1}$.

GREEDY MAP

Suppose that (A, q) satisfies the Pedicini condition with $a_0 = 0$. The greedy expansion can also be obtained by iterating the *greedy map corresponding to* (A, q) $T : J_{A,q} \to J_{A,q}$, defined by

$$T(x) = \begin{cases} qx - a_j, & x \in D(a_j) := \left[\dfrac{a_j}{q}, \dfrac{a_{j+1}}{q}\right), \quad 0 \le j < m, \\ qx - a_m, & x \in D(a_m) := \left[\dfrac{a_m}{q}, \dfrac{a_m}{q-1}\right]. \end{cases}$$

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If (b_i) is the greedy expansion of x in base q, then $b_n = j$ if and only if $T^{n-1}(x) \in D(a_j)$, $n \ge 1$, $a_j \in A$.

NORMALIZED ERRORS OF AN EXPANSION

The *normalized errors* of an expansion $(c_i) \in A^{\mathbb{N}}$ of $x \in J_{A,q}$ are defined by

$$\theta_n((c_i)) := q^n \left(x - \sum_{i=1}^{n} \frac{c_i}{q^i} \right), \quad n \in \mathbb{N}.$$

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Note that only the greedy expansion can be optimal.

MAIN RESULT

Till further notice, $A = \{0, 1\}$ and $q \in (1, 2)$. Such couples (A, q) satisfy the Pedicini condition, so each $x \in [0, 1/(q-1)]$ has a (greedy) expansion.

MAIN RESULT

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Let P be the set of bases $q \in (1, 2)$ satisfying one of the equalities

$$1 = \frac{1}{q} + \cdots + \frac{1}{q^n}, \quad n \geq 2,$$

and let $\mathcal{O}_q$ be the set of numbers with an optimal expansion in base q.

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Theorem

- *If $q \in P$, then $\mathcal{O}_q = [0, 1/(q-1)]$.*
- *If $q \in (1, 2) \setminus P$, then $\mathcal{O}_q$ is nowhere dense and has Hausdorff dimension less than 1.*

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- Let $\mathcal{U}_q$ be the set of of numbers in $[0, 1/(q-1)]$ with only one expansion in base q. Clearly, $\mathcal{U}_q \subseteq \mathcal{O}_q$.

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- It can be shown that $\mathcal{O}_q$ always contains the closure of $\mathcal{U}_q$.
- The greedy expansions $(0)^{n-1}1(0)^\infty$ and $(1)^n(0)^\infty$ of the numbers

$$\frac{1}{q^n} \quad \text{and} \quad \frac{1}{q} + \cdots + \frac{1}{q^n},$$

respectively, are optimal in base q, for each $n \geq 1$.

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The maximality of k implies that $x_n \notin \overline{\mathcal{U}_q}$. Hence $\mathcal{O}_q \setminus \overline{\mathcal{U}_q}$ is infinite for each $q \in (1,2)$.

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The greedy expansion of x is thus not optimal.

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$\theta_{n+2}((c_i))<\theta_{n+2}((b_i))$, so (b_i) is not optimal.

GREEDY MAP T_k

Each $x \in [0, 1/(q-1)]$ has an expansion in base q, so each such x has an expansion in base q^k $\quad (k \geq 1)$ with respect to the alphabet

$$A_k := \{c_1 q^{k-1} + c_2 q^{k-2} + \cdots + c_{k-1} q + c_k | c_1, \ldots, c_k \in \{0, 1\}\}.$$

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This implies that each $x \in J_{A,q} = J_{A_k, q^k}$ has a greedy expansion with respect to (A_k, q^k), sometimes called the *k-block greedy expansion*.

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We sometimes refer to the greedy map T_k corresponding to (A_k, q^k) as the *k-block greedy map*. The *k-fold greedy map T^k* is just the k-fold composition of the ordinary greedy map T.

COMPARING T_k WITH T^k

Note: The first digit of the greedy expansion of x with respect to (A_k, q^k) is not necessarily $b_1 q^{k-1} + \cdots b_{k-1} q + b_k$ if (b_i) is the greedy expansion of x with respect to (A, q):

Example

If $1 < q < G$ and $x = q^{-2} + q^{-3}$, then $q^{-1} < q^{-2} + q^{-3}$ whence $T_3(x) = 0 < T^3(x) = q^3 \left(x - q^{-1}\right)$.

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Proposition

- *For all $x \in [0, 1/(q-1)]$ and $k \geq 1$ we have $T_k(x) \leq T^k(x)$.*
- *$T_k(x) = T^k(x)$ for each $k \geq 1$ if and only if x has an optimal expansion.*

CRITERIA FOR $T_k = T^k$

Given $k \geq 1$, let $S_{q,k}$ be the set of all blocks $(c_1, \ldots, c_k) \in \{0,1\}^k = A^k$ such that

$$\sum_{i=1}^{k} \frac{d_i}{q^i} \neq \sum_{i=1}^{k} \frac{c_i}{q^i}$$

for each $(d_1, \ldots, d_k) \in A^k$ satisfying $(d_1, \ldots, d_k) > (c_1, \ldots, c_k)$.

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Note that $S_{q,k} \supseteq \{(b_1(x), \dots, b_k(x)) \mid x \in [0, 1/(q-1)]\}$, but in general we have no equality here:

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If $1 < q < G$, then $q^{-1} < q^{-2} + q^{-3}$, hence 011 belongs to $S_{q,3}$ but no greedy expansion starts with 011.

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Let the injective map $f : S_{q,k} \to [0, 1/(q-1)]$ be given by

$$f((c_1, \ldots, c_k)) = \frac{c_1}{q} + \cdots + \frac{c_k}{q^k}, \quad (c_1, \ldots, c_k) \in S_{q,k}.$$

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- $T_k = T^k$.
- $S_{q,k} = \{(b_1(x), \dots, b_k(x)) \mid x \in [0, 1/(q-1)]\}$.

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First we show that if $1 = q^{-1} + \cdots + q^{-n}$ for some n, then the greedy expansion of each $x \in [0, 1/(q-1)]$ is optimal.

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Suppose that $(c_1, \ldots, c_k)$ and $(d_1, \ldots, d_k)$ both belong to $S_{q,k}$ and that $(c_1, \ldots, c_k) > (d_1, \ldots, d_k)$. We must show that

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Let j be the first index for which $c_j = 1 > d_j = 0$. It is enough to show that

$$\sum_{i=1}^{k-j} \frac{d_{j+i}}{q^i} < 1.$$

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Blocks in $S_{q,k}$ cannot contain the subblock $0(1)^n$ because the sum corresponding to a block $0(1)^n$ equals the sum corresponding to the lexicographically larger block $1(0)^n$.

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We may conclude that $T_k = T^k$ for each $k \geq 1$; therefore each $x \in [0, 1/(q-1)]$ has an optimal expansion.

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More precisely, we saw that if $1 = q_n^{-1} + \cdots q_n^{-n}$ ($n \in \mathbb{N}$) and $q \in (q_n, q_{n+1})$, then no number belonging to the interval

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For each $q \in (1,2) \setminus P$ it is possible to construct an open subinterval of $[0,1)$ that does not meet $\mathcal{O}_q$.

OPTIMALITY IF THE BASE BELONGS TO $(1,2) \setminus P$

Let V be such an open interval. It is not hard to prove that the set

$$W := \{x \in [0, 1/(q-1)] \mid T^k(x) \notin V \text{ for all } k \geq 0\}$$

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We may conclude that $\mathcal{O}_q$ is nowhere dense and has Hausdorff dimension less than one if $q \in (1,2) \setminus P$.

GENERALIZATION

For a positive integer m, let $A = \{0, 1, \ldots, m\}$ and let $q \in (m, m+1)$. Note that the couple (A, q) satisfies the Pedicini condition.
Let P_m be the set consisting of bases $q \in (m, m+1)$ which satisfy one of the equalities

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- *If $q \in P_m$, then $\mathcal{O}_q = [0, m/(q-1)]$.*
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Note: Numbers belonging to $P = \bigcup_m P_m$ are sometimes called *confluent Parry numbers*.

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$$h_q(x) = \frac{1}{F(q)} \sum_{n=0}^{\infty} \frac{1}{q^n} \cdot 1_{[0,T^n(1))}(x),$$

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(*Lasota, Yorke*) For each positive integer k, the greedy map T_k corresponding to (A_k, q^k) is also ergodic with respect to a unique normalized a.c.i.m μ_k, as follows from a more general theorem on piecewise lineair expanding maps.

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$q \in P$ if and only if $\mu_1 = \mu_k$ for all $k \geq 1$.

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Sketch of proof: If $q \in P$ and $k \geq 1$, then μ_1 is also an a.c.i.m for T_k because $T_k = T^k$. Since the a.c.i.m is unique, it follows that $\mu_1 = \mu_k$. If $q \notin P$, we have already seen that $T_k < T^k$ for some k on some open subinterval of $[0, 1)$. Arguing by contradiction, one easily derives that $\mu_1 \neq \mu_k$ for this k.

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Open problem: Determine an explicit formula for the density of μ_k if $q \notin P$.

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Proposition

The set $\mathcal{O}_q$ is closed from above: if $x \notin \mathcal{O}_q$, then there exists a number $\delta = \delta(x) > 0$ so that $[x, x + \delta) \cap \mathcal{O}_q = \varnothing$.

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Proof.

If the greedy expansion (b_i) of x is not optimal, then there exists an expansion (c_i) of x and a number $n \geq 1$ such that

$$\sum_{i=1}^{n} \frac{b_i}{q^i} < \sum_{i=1}^{n} \frac{c_i}{q^i} \leq x.$$

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Hence x belongs to the interval

$$E := \left[\sum_{i=1}^{n} \frac{c_i}{q^i}, \sum_{i=1}^{n} \frac{c_i}{q^i} + \sum_{i=n+1}^{\infty} \frac{m}{q^i} \right).$$

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The map $x \mapsto (b_i)$ is continuous from the right, hence there exists an interval $[x, x+\delta)$ contained in E so that each number in $[x, x+\delta)$ has an expansion starting with $b_1 \dots b_n$ and one starting with $c_1 \dots c_n$. $\square$

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