Mean field models for neural networks with excitatory interactions

CRM Conference (Pisa)

François Delarue (Nice – J.-A. Dieudonné)

September 13-15 2017

Based on joint works with J. Inglis, S. Rubenthaler and E. Tanré

Part I. Motivation

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a. General picture

Basic purpose

- Provide a simple model for a neuronal network
  - with similar neurons

    $\rightsquigarrow$ focus on one single typical neuron
  - choose a standard model for the dynamics of the typical neuron

    $\rightsquigarrow$ examples: $\boxed{\text{diffusion process}}$ (with hard threshold),
jump processes (with soft threshold)
- Use mean field assumption to describe interactions
  - a neuron sees the others through the whole collectivity
  - global quantity of interest $\leadsto$ $\boxed{\text{global averaged firing rate}}$

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  - global quantity of interest $\leadsto$ $\boxed{\text{global averaged firing rate}}$
- $\boxed{\text{Excitatory}}$ feature
  - if global averaged firing rate $\uparrow \Rightarrow$ each neuron is more likely to spike
  - would make sense to regard inhibitory counterpart

Challenges

- Mean field limit

  ◦ derive the limit model as the number of neurons $\uparrow$

  ◦ expect $\boxed{\text{propagation of chaos / LLN}}$

  ◦ reduce the asymptotic analysis to one typical neuron with interaction with theoretical distribution?

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  ◦ existence and uniqueness of solutions to $\boxed{\text{asymptotic model}}$? influence of the excitation?

  ◦ prove convergence of finite models

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- Literature

  $\rightsquigarrow$ mean field integrate and fire [Lewis and Rinzel (03); Ostojic, Brunel and Hakim (09); Caceres, Carrillo, Perthame (11,14); DIRT; Inglis and Talay (16)]

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  $\rightsquigarrow$ application to systemic risk [Hambly and Ledger (16), Nadotchiy and Shkolnikov (17)]

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  $\rightsquigarrow$ models without hard threshold [Fournier Löcherbach (16)], Hawks model of mean field types [Chevallier (16)]

Part I. Motivation

b. A general form for the finite network

General LIF model for a single neuron

- Describe membrane potential of the neuron

  $\rightsquigarrow$ neuron fires if membrane potential is high

  - several simple models

  $\rightsquigarrow$ jump model with soft threshold $\rightsquigarrow$ spike is more likely if potential is high

  $\rightsquigarrow$ diffusive model with $\boxed{\text{hard threshold}}$ $\rightsquigarrow$ spike occurs if potential reaches a threshold

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- Subthreshold dynamics

$$\frac{d}{dt}V_t = -\lambda V_t + I_t^{\text{int}} + I_t^{\text{ext}}$$

  - λ connected with properties of the membrane
  - I^{int} $\rightsquigarrow$ current due to interactions with other cells
  - I^{ext} $\rightsquigarrow$ collective effect due to external phenomena

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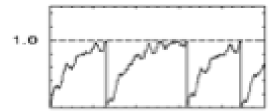
- Threshold $\rightsquigarrow$ spike whenever V reaches firing value V_F

$$\tau = \inf\{t \geq 0 : V_t \geq V_F\}$$

  - after τ (no refractory period) $\rightsquigarrow$ reset potential at $V_\tau = V_R$



Taken from W. Gerstner and W. Kistler, Spiking neuron models 2002

Currents for connected neurons

- Label the neurons $i = 1, \ldots, N$

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- External current

$$\boxed{I_t^{\mathrm{ext},i} = \text{mean-trend}_t^i + \text{noise}_t^i}$$

  - focus on the noise $\rightsquigarrow$ $\text{noise}_t^i = (\dot{W}_t^i)_{t \geq 0}$ white noise
  - very strong assumption $\rightsquigarrow$ start with independent noises
  - may think of correlated cases as well $\rightsquigarrow$ more complicated [HL]

Mean-field interaction

- Force symmetric interactions (no privileged interactions)
  - $I_t^{\text{int}}(V_t^j, j \neq i)$ depending on the empirical distribution

$$I_t^{\text{int}}(V^j, j \neq i) = I_t^{\text{int}}\Big(N^{-1} \sum_{j \neq i} \delta_{V_\cdot^j}\Big)$$

- Subthreshold dynamics

$$dV_t^i = -\lambda V_t^i dt + I_t^{\text{int}}\Big(N^{-1} \sum_{j \neq i} \delta_{V_\cdot^j}\Big) dt + dW_t^i$$

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- Typical neuron interacts with its own law $\rightsquigarrow$ McKean-Vlasov eq.

$$dV_t = -\lambda V_t dt + I_t^{\text{int}}(\mathcal{L}(V_t)) dt + dW_t$$

Part I. Motivation

c. Examples

Choice of the interaction functional

- Frequently used model ([BH, IT])
  - $I_t^{\mathrm{int}}\Big(N^{-1}\sum_{j\neq i}\delta_{V^j}\Big)$ based on mean activity of the network

    $\rightsquigarrow I_t^{\mathrm{int}}\Big(N^{-1}\sum_{j\neq i}\delta_{V^j}\Big)$ function of $\frac{1}{N}\sharp\{\text{spikes} \leq t\}$

    $\rightsquigarrow$ if function is $\begin{cases}\uparrow & \Rightarrow \quad \text{excitation} \\ \downarrow & \quad\quad\ \text{inhibition}\end{cases}$
- Other version (see [OBH, DIRT, NS]) $\rightsquigarrow$ interactions
  - replace interaction currents by interaction pulses

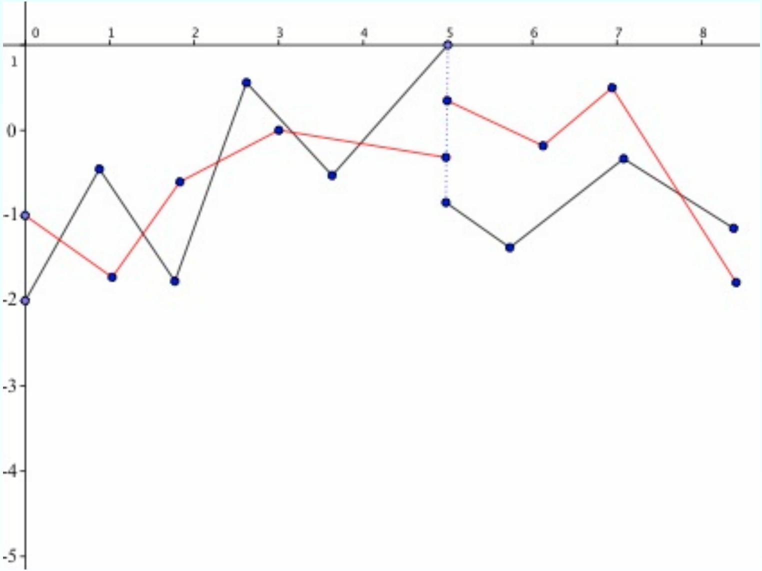
$$I_t^{\mathrm{int}}(V^j, j\neq i) = \frac{d}{dt}\frac{\alpha}{N}\sum_{j\neq i}\mathbf{1}_{\{V^j_{t-}=V_F\}}$$

$$= \frac{d}{dt}\frac{\alpha}{N}\sharp\{\text{spiking neurons } \neq i \text{ at } t\}$$

  - $\alpha > 0 \Leftrightarrow$ instantaneous self-excitatory interaction
- Replace spikes by defaults $\rightsquigarrow$ systemic risk in economy [BH,NS]

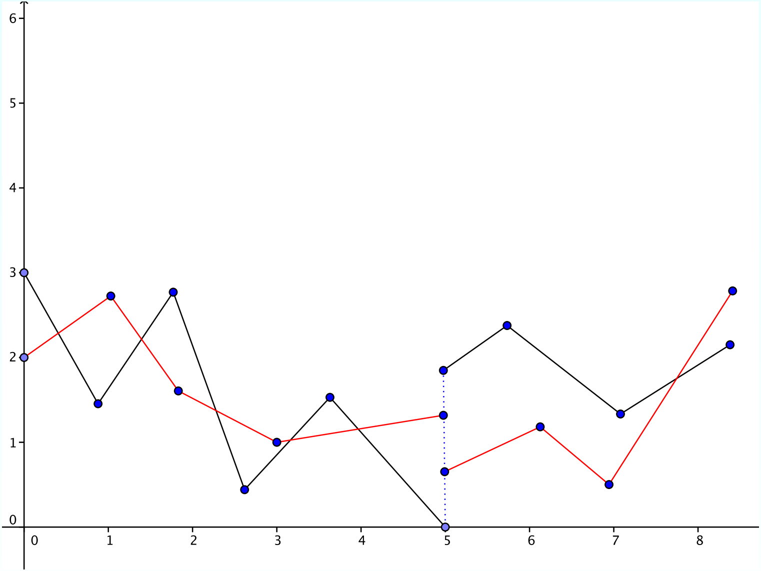
Picture for neuronal model

- For $V_F = 1$ and $V_R = -1$ threshold is zero



Picture for systemic risk model

- Consider V_F minus the potential $\rightsquigarrow$ threshold is zero



Part II. Limiting model

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a. Standard McKV equations

A non-singular particle system

- $\boxed{\text{Forget}}$ the spikes and focus on standard dynamics

$$dX_t^i = b(X_t^i, \bar{\mu}_t^N)dt + \sigma(X_t^i, \bar{\mu}_t^N)dW_t^i$$

$\circ$ $X_0^1, \ldots, X_N^i$ i.i.d. (and $\perp\!\!\!\perp$ of noises), $\quad \bar{\mu}_t^N = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \delta_{X_t^i}$

- $\exists!$ if the coefficients are Lipschitz in all the variables $\rightsquigarrow$ need a suitable distance on space of measures

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- Use the Wasserstein distance on $\mathcal{P}_2(\mathbb{R}^d)$

$$\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d), \quad W_2(\mu,\nu) = \left(\inf_\pi \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\pi(x,y)\right)^{1/2},$$

where π has μ and ν as marginals on $\mathbb{R}^d \times \mathbb{R}^d$

  - X and X' two r.v.'s $\Rightarrow W_2(\mathcal{L}(X), \mathcal{L}(X')) \le \mathbb{E}[|X-X'|^2]^{1/2}$

- Example $W_2\left(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}, \frac{1}{N}\sum_{i=1}^N \delta_{x_i'}\right) \le \left(\frac{1}{N}\sum_{i=1}^N |x_i - x_i'|^2\right)^{1/2}$

McKean-Vlasov SDE

- Expect some $\boxed{\text{decorrelation / averaging}}$ in the system as $N \uparrow \infty$
  - replace the empirical measure by the theoretical law

$$dX_t = b(X_t, \mathcal{L}(X_t))dt + \sigma(X_t, \mathcal{L}(X_t))dW_t$$

- Cauchy-Lipschitz theory
  - assume b and σ Lipschitz continuous on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \Rightarrow$ unique solution for any given initial condition in L^2
  - proof works as in the standard case taking advantage of

$$\mathbb{E}\Big[\big|(b,\sigma)(X_t, \mathcal{L}(X_t)) - (b,\sigma)(X'_t, \mathcal{L}(X'_t))\big|^2\Big] \leq C\mathbb{E}[|X_t - X'_t|^2]$$

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- Propagation of chaos
  - each $(X_t^i)_{0\leq t\leq T}$ converges in law to the solution of MKV SDE
  - particles get $\boxed{\text{independent}}$ in the limit $\rightsquigarrow$ for k fixed:

$$(X_t^1, \ldots, X_t^k)_{0\leq t\leq T} \xrightarrow[\mathcal{L}]{} \mathcal{L}(\text{MKV})^{\otimes k} = \mathcal{L}((X_t)_{0\leq t\leq T})^{\otimes k} \quad \text{as } N \nearrow \infty$$

  - $\lim_{N\nearrow\infty} \sup_{0\leq t\leq T} \mathbb{E}[(W_2(\bar{\mu}_t^N, \mathcal{L}(X_t))^2] = 0$

Part II. Limiting model

b. Formulation of the asymptotic problem

Ansatz

- Recall the subthreshold dynamics of the finite network

$$V_t^i = V_0^i - \lambda \int_0^t V_s^i ds + \frac{\alpha}{N} \sum_{j \neq i} \sharp\{\text{neuron}(j) \text{ spiked before } t\} + W_t^i$$

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- Heuristics $\rightsquigarrow$ same formal reasoning as for a regular interaction current

$$\boxed{I_t^{\text{int}}(V^j, j \neq i) \underset{N \to +\infty}{\sim} \alpha \mathbb{E}(M_t)}$$

  - M_t = number of spikes for typical neuron up to t

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- Subthreshold dynamics for typical neuron as $N \to \infty$

$$V_t = V_0 - \lambda \int_0^t V_s ds + \alpha \mathbb{E}(M_t) + W_t$$

  - $M_t = \sharp\{t \geq 0 : V_{t-} = V_F\}$ depends on V!

- Typical non-singular interactions $\int_0^t b(\mathbb{E}(M_s))ds$ [BH,IT]; see also MFG [Campi,Fischer]

Interpretation of the mean-field interaction

- Subthreshold dynamics

$$V_t = V_0 - \lambda \int_0^t V_s ds + \alpha \mathbb{E}(M_t) + W_t$$

  - firing value $V_F = 1$, reset (after spiking) $V_R = 0$

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- Crucial question: what class of $\boxed{\text{admissible solutions}}$?

  - class of solutions dictates regularity for $\mathbb{E}(M_t) \rightsquigarrow$ physical interpretation?

$$\begin{aligned} &\mathbb{E}(M_{t+h} - M_t) \\ &\sim_{N=\infty} \text{probability of spike in } [t, t+h] \\ &\sim_{N<\infty} \text{proportion of spikes in } [t, t+h] \end{aligned}$$

  - $\mathbb{E}(M_t)$ is allowed to jump $\leftrightsquigarrow$ large proportion of neurons may spike at the same time

  - may stand for massive simultaneous spikes in the system

Instantaneous firing rate

- Meaning for requiring $e : t \mapsto \mathbb{E}(M_t)$ to be differentiable?

$$\text{probability of spike in } [t, t+h] \sim e'(t)h$$

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- Subthreshold dynamics if differentiability

$$dV_t = -\lambda V_t dt - \alpha e'(t) dt + dW_t$$

  - SDE $\rightsquigarrow$ stochastic calculus and regularizing effect
  - $\mathbb{P}(V_t \in dy) = p(t, y) dy, \quad t > 0, \quad y < 1$

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- $\boxed{\text{Fokker Planck equation}}$

$$\partial_t p(t, y) + \partial_y[(-\lambda y + \alpha e'(t))p(t, y)] - \frac{1}{2}\partial^2_{yy} p(t, y) = e'(t)\delta_0$$

  - $p(t, 1) = 0$ and $\partial_y p(t, 1) = -\frac{1}{2}e'(t)$
  - control of e' $\Leftrightarrow$ control of the mass near 1

Part II. Limiting model

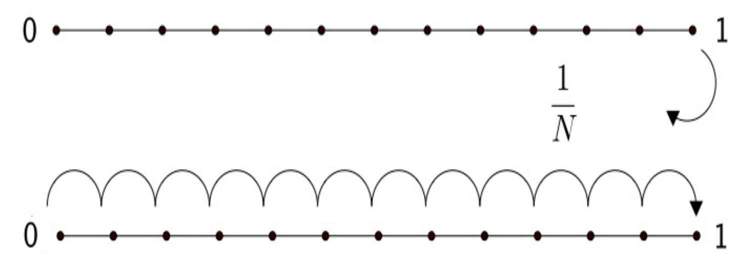
c. The need for $\alpha < 1$

Cascade phenomenon

- Difficulty α will dictate the smoothness of e! Cascade phenomenon in the modeling if $\alpha > 1$!

- Example: runaway behavior if reset ($V_R = 0$, $V_F = 1$) $\rightsquigarrow$ plot V_F-potential

  ◦ choose $N + 1$ neurons, $\alpha = (N + 1)/N$ and $V_0^i = i/N$, $i = 0, \ldots, N$,



  ◦ particles keep jumping!

  ◦ $\alpha < (N + 1)/N \Rightarrow$ no way for defaulting twice at same time

Reformulating the limiting model

- Convenient to reformulate solutions $\rightsquigarrow$ unknown without reset

$$\boxed{Z_t = V_t + M_t}$$

  - $M_t = \sharp$ different positive integers crossed by $(Z_s)_{0\leq s\leq t}$

$$M_t = \left\lfloor \big(\sup_{0\leq s\leq t} Z_s\big)_+ \right\rfloor$$

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- Dynamics of $(Z_t)_{t \geq 0}$

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- $\boxed{\text{Application}}$

$$\big(\sup_{0 \le s \le t} Z_s \big)_+$$
$$\le (Z_0)_+ + 2|\lambda| \int_0^t \big(\sup_{0 \le r \le s} Z_r \big)_+ ds + \alpha \mathbb{E}\Big[\big(\sup_{0 \le s \le t} Z_s \big)_+ \Big] + \sup_{0 \le s \le t} |W_s|$$

  - $\alpha < 1$ needed to get an *a priori* bound

Part II. Limiting model

c. Solvability results

Solvability of the regular model

- Existence of regular solutions in arbitrary time?
  - avoid blow-up of e' in finite time?
  - $\Leftrightarrow$ avoid massive spikes?

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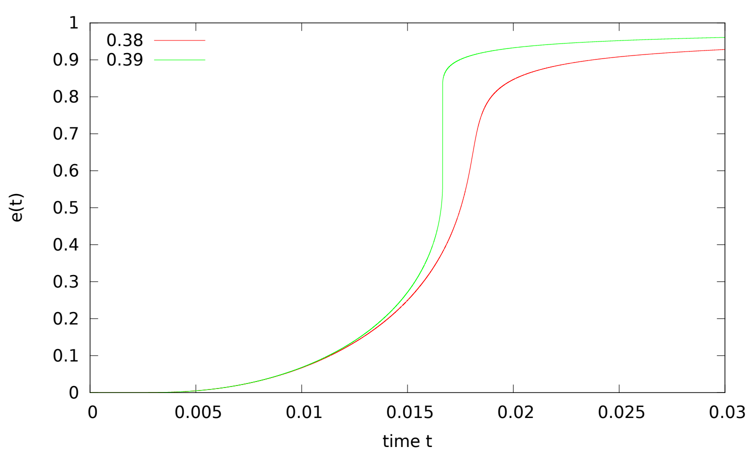
  - explicit (but non-optimal) bounds on critical values α
- Brownian example: $\lambda = 0$ and $V_0 = 0.8$ ($V_F = 1$, $V_R = 0$)
  - existence of regular solutions if $\alpha \leq 0.10$
  - no regular solutions if $\alpha \geq 0.54$
  - numerically, critical value $\sim 0.38\ldots$
- Exemple O-U $\lambda \to \infty \Rightarrow$ critical $\alpha \to 1$ ($\Leftrightarrow \lambda$ fixed and $\sigma \to 0$)

Illustration



- Need a general notion of solutions with blow-up
  - existence is known [DIRT], uniqueness is partial only [NS]

Lower bound for criticality

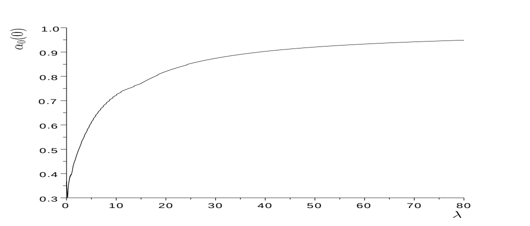


Figure: Plot of $\alpha_0(0)$ in terms of $\lambda \in [0; 80]$.

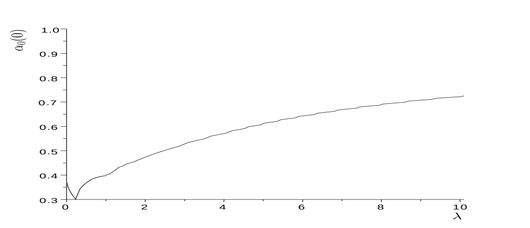


Figure: Plot of $\alpha_0(0)$ in terms of $\lambda \in [0; 10]$.

Part II. Limiting model

e. Existence of a blow-up for $\alpha \gg 0$

Caceres Carrillo Perthame argument

- Choose $V_0 = v_0$ and $\lambda = 0$ to simplify
- Compute $\boxed{\text{Laplace transform of potential}}$

$$z(t) = \mathbb{E}[\exp(\mu V_t)], \quad \text{for } \mu > 0$$

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  - provided $e(t) = \mathbb{E}[M_t]$ is differentiable $\rightsquigarrow$ Itô's formula yields

$$\frac{d}{dt} z(t) = \underbrace{\left[\alpha\mu e'(t) + \frac{\mu^2}{2}\right]}_{\eta(t)} z(t) + \underbrace{[1 - \exp(\mu)] e'(t)}_{\nu(t)}$$

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◦ solve the ODE and use $\boxed{z(t) \leq \exp(\mu)}$

$$0 \leq \exp\left(-\mu + \int_0^t \eta(s) ds\right)\left[z(0) - \int_0^t \nu(s) \exp\left(-\int_0^s \eta(u) du\right) ds\right] \leq 1$$

$\rightsquigarrow$ let t tend to ∞

Caceres Carrillo Perthame argument

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$$0 = z(0) - \int_0^\infty \nu(s) \exp\left(-\int_0^s \eta(u) du\right) ds$$

  - integrate explicitly

$$1 - \frac{\alpha\mu \exp(\mu v_0)}{\exp(\mu) - 1} = \frac{\mu^2}{2} \int_0^\infty \exp\left(-\alpha\mu e(s) - \frac{\mu^2}{2} s\right) ds \geq 0$$

Part III. Solving the model for $\alpha \ll 1$

a. General plan

Sketch of the proof

- Difficulty: competition between noise and mean-field

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- Typical scheme for nonlinear models
  - existence and uniqueness in $\boxed{\text{short time}}$ on $[0, T^\star]$

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  - if $\quad \frac{1}{dy}\mathbb{P}(V_0 \in dy) \leq \beta(1-y) \quad$ for $y \in (1-\varepsilon, 1)$

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$$e \in C^1([0, T])$$

  - where $dV_t = -\lambda V_t dt + \alpha e'(t)dt + dW_t$ before spike

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Ingredients for the contraction in small time

- $\boxed{\text{Fix } e}$ and consider $dV_t^e = -\lambda V_t^e dt + \alpha e'(t) dt + dW_t$ before spike
  - $\tau_k^e = k^{\text{th}}$ hitting time of 1

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- Use first Markov property to refresh after reset

$$\Gamma(e)(t) = \sum_{k\geq 1} \int_0^t \mathbb{P}(\tau_1^{e^{\sharp_s}} \leq t - s | X_0^{e^{\sharp_s}} = 0)\mathbb{P}(\tau_k^e \in ds) + \mathbb{P}(\tau_1^e \leq t)$$

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- $\boxed{\text{Use Fokker-Planck}}$ for $p^e(t, y) = \frac{d}{dy}\mathbb{P}(V_t^e \in dy, t < \tau_1)$
  - $\partial_t p^e(t, y) + \partial_y[(-\lambda y + \alpha e'(t))p^e(t, y)] - \frac{1}{2}\partial_{yy}^2 p^e(t, y) = 0$

    $\rightsquigarrow p^e(t, 1) = 0$
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- $\boxed{\text{Use parametrix}}$ when $V_0 = v_0 < 1$

$$p^e(t,y) = q(t, v_0, y) - \int_0^t \int_{-\infty}^1 (\alpha e'(s) - \lambda)\, \partial_z pe(s,z) q(t-s, z, y) dz ds$$

$\rightsquigarrow q = p^0$ for $e = 0$ and drit $-\lambda(y-1)$

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  - use $p^e(0,y) \leq \beta(1-y)$ to control $\partial_z p^e(s,z)$

Part III. Solving the model for $\alpha \ll 1$

b. From small to arbitrary time

Sketch of the proof

- Difficulty: competition between noise and mean-field

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Scheme for the a priori estimate

- Assume $\exists$ solution with $e \in C^1$ on $[0, T]$
  - where $dV_t = b(V_t)dt + \alpha e'(t)dt + dW_t$ before spike
  - with reset after spike

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  - bound for $p(t,y) = \mathbb{P}(V_t \in dy)/dy$
- Bound of $p(t,y)$
  - rough bound using (non-killed) Gaussian kernels

$$V_0 > \varepsilon \Rightarrow p(t,y) \leq C(\varepsilon,\alpha), \quad y \in (0, \varepsilon/4)$$

  - very bad (can't see $p(t,1) = 0$) but explicit

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- Four steps to get $\frac{1}{dy}\mathbb{P}(V_0 \in dy) \leq \beta(1-y)$
  - bound for $p(t, y) = \mathbb{P}(V_t \in dy)/dy$
  - $1/2$ Hölder bound for e
  - Hölder regularity of $p(t, y)$ in y
  - Lipschitz regularity of $p(t, y)$ in y
- Bound of $p(t, y)$
  - rough bound using (non-killed) Gaussian kernels

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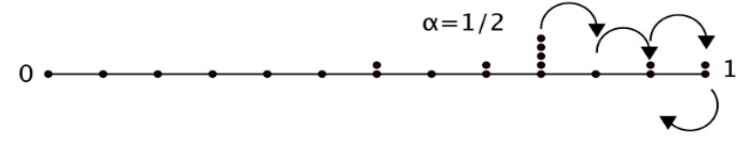
Part III. Solving the model for $\alpha \ll 1$

c. Implementing the rough bound for p

Continuity of e

- Condition for continuity of e?



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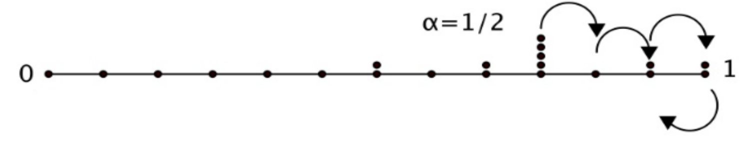

$$\Delta e(t) = e(t) - e(t-) = 0$$
$$\Leftrightarrow \exists \delta_n \downarrow 0 : \underbrace{\text{kick due to particles in } [0, \delta_n)}_{\alpha \int_0^{\delta_n} p(t-, y) dy} < \delta_n$$

- ∘ if $p(t, y) < 1/\alpha$ for $y \in [0, \varepsilon)$ then $e(t) = e(t-)$

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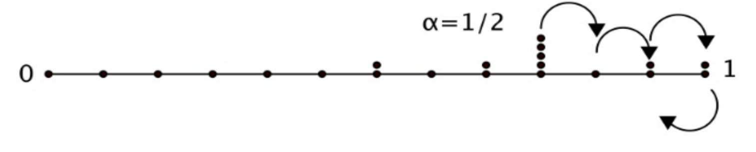

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- ∘ if $p(t, y) < 1/\alpha$ for $y \in [0, \varepsilon)$ then $e(t) = e(t-)$

- Application $\Rightarrow$ implement the bound $p(t, y) \leq C(\varepsilon, \alpha)$
  - ∘ if $C(\varepsilon, \alpha)\alpha < 1$ then continuity of e
  - ∘ $\boxed{\text{provides the condition } \alpha \text{ small!}}$
  - ∘ continuity dictated by Brownian: e 1/2-Hölder

Regularity of p close to the boundary

- Recall Dirichlet condition $p(t,1) = 0$
  - p satisfies Fokker-Planck $\rightsquigarrow$ Feynman-Kac

$$p(T,y) = \mathbb{E}\Big[p(T-\rho, Y_\rho)\exp(\lambda\rho)\Big|Y_0 = y\Big]$$

  - where $dY_t = \lambda Y_t dt - \alpha e'(T-t)dt + dW_t$
  - $\rho = \inf\{t \geq 0 : Y_t \notin (1-\delta, 1)\} \wedge T$ (don't see the reset)

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  - $\rho = \inf\{t \geq 0 : Y_t \notin (1-\delta, 1)\} \wedge T$ (don't see the reset)
- Regularity of p at the boundary $\leftrightsquigarrow \mathbb{P}\{Y_\rho = 1\}$

$$p(T,y) \leq C\mathbb{P}(\{Y_\rho = 1-\delta\} \cup \{\rho = T\}) \sup_{t\in[0,T], x\in[1-\delta,1]} p(t,x)$$

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  - competition between B and e

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- Probability to hit the boundary
  - competition between B and e

    $\rightsquigarrow$ e pushes Y away from 1
  - e 1/2 Hölder $\Rightarrow$ B wins with >0 probability
  - $y > 1-\delta/2$ and $\delta \ll 1 \Rightarrow p(T, y) \leq (1-c)\sup_{t\in[0,T], x\geq 1-\delta} p(t, x)$

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- Probability to hit the boundary
  - competition between B and e

    $\rightsquigarrow$ e pushes Y away from 1
  - e 1/2 Hölder $\Rightarrow$ B wins with >0 probability
  - get Hölder decay and then Lipschitz (barrier lemma)

Part IV. Solutions with blow-up

a. Physical solutions of the particle system

Returning to the particle system

- Specify mean field interaction

$$V_t^{i,N} = V_0^{i,N} - \lambda \int_0^t V_s^{i,N} ds + \frac{\alpha}{N} \sum_{j=1}^{N} M_t^{j,N} + W_t^i - M_t^{i,N}$$

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- Not well-posed! take $N = 3$ and
  - $t : M_{t-}^1 = M_{t-}^2 = M_{t-}^3 = 0,\ V_{t-}^1 = 1,\ V_{t-}^2, V_{t-}^3 \in (1 - \frac{2\alpha}{3}, 1 - \frac{\alpha}{3})$

    $\rightsquigarrow$ 1st solution $M_t^1 = 1,\ M_t^2 = M_t^3 = 0$ kick $= \frac{\alpha}{3}$

    $\rightsquigarrow$ 2nd solution $M_t^1 = M_t^2 = M_t^3 = 1$ kick $= 1$

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- Previous counter-example $\rightsquigarrow$ need to $\boxed{\text{order}}$ jumps
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$$\Gamma_{k+1} = \left\{ i \in \{1, \dots, N\} \backslash \Gamma_0 \cup \dots \cup \Gamma_k : X^i_{t-} + \alpha \frac{|\Gamma_0 \cup \dots \cup \Gamma_k|}{N} \geq 1 \right\}$$

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- Global set of particles that spike $\rightsquigarrow \Gamma = \bigcup_{0 \leq k \leq N-1} \Gamma_k$

$$V^i_t = V^i_{t-} + \frac{\alpha|\Gamma|}{N} \text{ if } i \notin \Gamma, \quad V^i_t = V^i_{t-} + \frac{\alpha|\Gamma|}{N} - 1 \text{ if } i \in \Gamma.$$

Part IV. Solutions with blow-up

b. Physical solutions of the limiting system

Rules for spiking

- Seek càd-làg solutions
- From particle system $\rightsquigarrow$ need to prescribe rules for spiking
  - no more than 1 spike at a given time $\Rightarrow \Delta M_t = M_t - M_{t-} \in \{0, 1\}$

$$\Delta\mathbb{E}[M_t] = \mathbb{P}[V_{t-} + \underbrace{\alpha\Delta\mathbb{E}[M_t]}_{\text{kick}} \geq 1]$$

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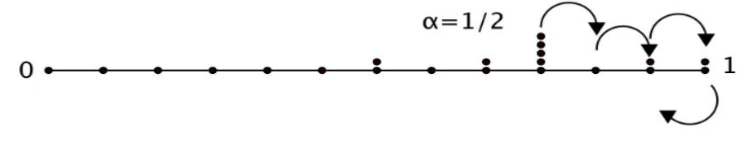

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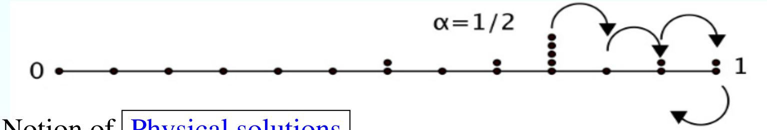

- Notion of $\boxed{\text{Physical solutions}}$
  - no jump if remaining mass after jump is too small!

$$\Delta e(t) = \inf\{\eta \geq 0 : \mathbb{P}(V_{t-} + \alpha\eta \geq 1) < \eta\}$$

Solutions with blow-up

- Description of the jumps of $e(t) = \mathbb{E}(M_t)$ when blow-up?

$$\begin{aligned}\Delta e(t) = e(t) - e(t-) \geq \delta_0 \\ \Leftrightarrow \forall \delta \leq \delta_0,\ \text{kick due to particles in } [0,\delta) \geq \delta \\ \Leftrightarrow \forall \delta \leq \delta_0,\ \underbrace{\alpha \int_0^\delta p(t-,y)dy}_{\text{kick due to particles in } [0,\delta)} \geq \delta\end{aligned}$$

  - restart with density $p(t,y) = p(t-, y + \Delta e(t))$ for y near 1
- Construction of a solution $\Rightarrow$ approximation
  - risk modeling $\rightsquigarrow$ massive/systemic default?
- Uniqueness?
  - [NS] : uniqueness as long as $t : \int_0^t |e'(s)|^2 ds < \infty$ for

Reformulation

- Convenient to reformulate solutions $\rightsquigarrow$ unknown without reset

$$\boxed{Z_t = V_t + M_t}$$

  - $M_t = \sharp$ different positive integers crossed by $(Z_s)_{0 \le s \le t}$

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- $\boxed{\text{Similar transformation}}$ with particle system

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Part V. Construction of solutions with blow up

a. M1 topology

Description

- Need convenient compactness for $\uparrow$ functions
  - rationale for using M1 (different from J1!)
    - $\rightsquigarrow$ topology on $\mathcal{D}([0,T],\mathbb{R})$, see [Skorohod,Whitt]

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- $\boxed{\text{Distance}}$ between f_1, f_2

$$d_{M_1}(f_1,f_2) = \inf_{(u_j,r_j),j=1,2} \{\|u_1 - u_2\|_\infty \vee \|r_1 - r_2\|_\infty\}$$

Compact sets

- Modulus of continuity

$$w_T(f,t,\delta) = \sup_{0\vee(t-\delta)\le t_1<t_2<t_3\le T\wedge(t+\delta)} \mathrm{dist}(f(t_2),[f(t_1),f(t_3)])$$

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- Connection with standard modulus
  - if $(f_n)_{n\ge 1}\to f \Rightarrow$ for any continuity point of f

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{s\in[0\vee(t-\delta),T\wedge(t+\delta)]}|f_n(s)-f(s)| = 0$$

Part V. Construction of solutions with blow up

b. Convergence of the particle system

Main Statement [DIRT]

- Recall particle system

$$Z_t^{i,N} = V_0^{i,N} - \lambda \int_0^t \left(Z_s^{i,N} - M_s^{i,N} \right) ds + \frac{\alpha}{N} \sum_{j=1}^{N} M_t^{j,N} + W_t^i$$

$$M_t^{i,N} = \lfloor (\sup_{s \in [0,t]} Z_s^{i,N})_+ \rfloor$$

  - $\boxed{\text{empirical measure}} \rightsquigarrow \bar{\mu}_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\tilde{Z}_\cdot^{i,N}}$
  - r.v. with values in $\mathcal{P}(\mathcal{D}([0,T],\mathbb{R})) \rightsquigarrow$ call Π_N the law of $\bar{\mu}_N$

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- Claim 2: For a weak limit Π_∞, for Π_∞-a.e. $\mu \in \mathcal{P}(\mathcal{D}([0,T],\mathbb{R}))$, the canonical process $(z_t)_{t\in[0,T]}$ generates, under μ, a $\boxed{\text{physical solution}}$
  - under μ, $(z_t - z_0 + \lambda \int_0^t (z_s - m_s) ds - \alpha \langle \mu, m_t \rangle)_{t\in[0,T)}$ is B.M.

$\rightsquigarrow m_t = \lfloor (\sup_{0\leq s\leq t} z_s)_+ \rfloor$ and $\langle \mu, m_t \rangle = \int m_t d\mu$

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  - rewrite in terms of canonical process under $\bar{\mu}_N$

$$\bar{\mu}_N \circ \left(z_t - z_0 + \lambda \int_0^t (z_s - m_s)ds - \alpha\langle\bar{\mu}_N, m_t\rangle\right)_{0\le t\le T}^{-1} = \mathbb{P} \circ \left(\frac{1}{N}\sum_{i=1}^N \delta_{W_\cdot^i}\right)^{-1}$$

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- Main difficulty : continuity of the functional $z \mapsto (\sup_{0\le s\le t} z_s)_+$
  - may be false! True if z really crosses threshold

Part V. Construction of solutions with blow up

c. Another construction

Delayed interaction

- Subthreshold potential with delayed interaction

$$V_t^\delta = V_0 - \lambda \int_0^t V_s^\delta ds + \alpha e_\delta(t) + W_t$$

  - $M_t^\delta = \sum_{k\geq 1} \mathbf{1}_{[0,t]}(\tau_k^\delta)$, $\quad \tau_k^\delta = \inf\left\{t > \tau_{k-1}^\delta : V_{t-}^\delta + \alpha \Delta e_\delta(t) \geq 1\right\}$
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Part V. Extensions

a. Model with common noise

Model with a common noise

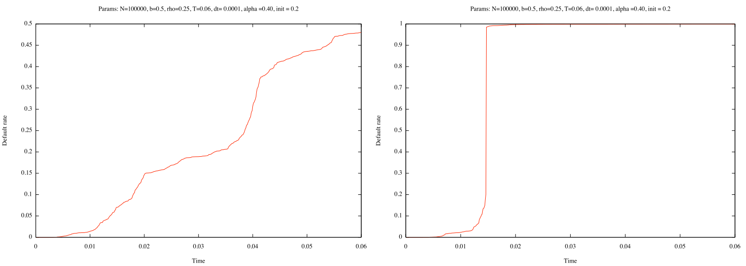
- Common source of noise in dynamics of the neurons

$$V_t^i = V_0^i - \lambda \int_0^t V_s^i ds + I_t^i + W_t^i + W_t^0$$

- Mean-field modeling

$$V_t = V_R - \lambda \int_0^t V_s ds + \alpha \mathbb{E}(M_t | W^0) + W_t + W_t^0$$

  - same $\alpha \rightsquigarrow$ two $\neq$ plots: competition with common noise



$\leadsto$ See Hambly, Ledger (without singular interactions)

Part V. Extensions

b. Model with random weights

Part V. Extensions

c. Model with spatial component