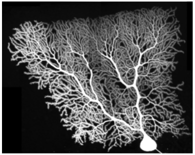
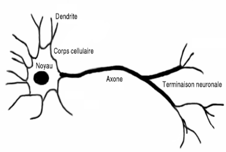
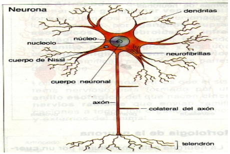
PDEs for neural networks

analysis and behaviour

Benoît Perthame







Plan of the course

I. The single neuron,

II. Networks (Wilson-Cowan, Integrate & Fire networks)

III. Networks, time elapsed models

III. 1. Desynchronisation

III. 2. Spontaneous activity

III. 3. Relation to I&F

Elapsed time structured model

Based on K. Pakdaman, J. Champagnat, J.-F. Vibert

- s represents the time elapsed since the last discharge
- $n(s,t)$ probability of finding a neuron in 'state' s at time t
- $N(t)$ = activity of the network

$$\frac{\partial n(s,t)}{\partial t} + \overbrace{\frac{\partial n(s,t)}{\partial s}}^{\text{elapsed time advances}} + \overbrace{r(s, bN(t))\; n(s,t)}^{\text{firing neurons}} = 0,$$

$$N(t) := n(s=0,t) = \underbrace{\int_0^{+\infty} r(s, bN(t))\; n(s,t) ds}_{\text{reset of firing neurons}},$$

The renewal equation

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + r(s)\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s)\ n(s,t) ds \end{cases}$$

We want a steady state (compute the growth rate)

Long standing equation (Feller, renewal process)

The renewal equation

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda + r(s)]\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s)\ n(s,t) ds \end{cases}$$

We want a steady state

$$\bar{n}(s) = \bar{n}(0) e^{-R(s)-\lambda s}, \qquad R(s) = \int_0^s r$$

In the boundary condition

$$\bar{n}(0) = \bar{n}(0) \int_0^\infty b(s) e^{-R(s)-\lambda s} ds$$

Find λ such that

$$\int_0^\infty b(s) e^{-R(s)-\lambda s} ds = 1$$

The renewal equation

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda + r(s)]\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s)\ n(s,t) ds \end{cases}$$

$$\bar{n}(s) = \bar{n}(0) e^{-R(s)-\lambda s}, \qquad R(s) = \int_0^s r$$

Find λ such that

$$\int_0^\infty b(s) e^{-R(s)-\lambda s} ds = 1$$

When $b(\cdot) = r(\cdot)$ then $\lambda = 0$

The renewal equation

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda + r(s)] \; n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s) \; n(s,t) ds \end{cases}$$

$$\bar{n}(s) = \bar{n}(0) e^{-R(s) - \lambda s}, \qquad \int_0^{\infty} b(s) e^{-R(s) - \lambda s} ds = 1$$

Prove that

$$n(s,t) \to \bar{n}(s)$$

Use Generalized Relative Entropy

The renewal equation

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda + r(s)]\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s)\ n(s,t) ds \end{cases}$$

$$\frac{\partial \phi(s)}{\partial s} - [\lambda + r(s)]\ \phi(s) = -\phi(0) b(s)$$

We have

$$\frac{d}{dt} \int n(s,t)\phi(s) = 0$$

Usually for a conservative equation $\phi \equiv 1$

For example, when $b(\cdot) = r(\cdot)$, $\lambda = 0$.

Then the relative entropy is $\int_0^{\infty} \bar{n}(s) H(\frac{n(s,t)}{\bar{n}(s)}) ds$

The renewal equation

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda + r(s)]\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s)\ n(s,t) ds \end{cases}$$

$$\frac{\partial \phi(s)}{\partial s} - [\lambda + r(s)]\ \phi(s) = -\phi(0) b(s)$$

The Generalized Relative Entropy is

$$E(t) := \int_0^{\infty} \phi(s) \bar{n}(s) H\left(\frac{n(s,t)}{\bar{n}(s)}\right) ds$$

The renewal equation

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda + r(s)]\; n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s)\; n(s,t) ds \end{cases}$$

$$\frac{\partial \phi(s)}{\partial s} - [\lambda + r(s)]\; \phi(s) = -\phi(0) b(s)$$

The Generalized Relative Entropy is

$$E(t) := \int_0^{\infty} \phi(s)\bar{n}(s) H(\frac{n(s,t)}{\bar{n}(s)}) ds$$

$$\frac{dE(t)}{dt} = -D_H(t)\phi(0)$$

$$D_H(t) = \int b(s)\bar{n}(s) H\left(\frac{n(s,t)}{\bar{n}(s)}\right) ds - H\Big(\int b(s)\bar{n}(s)\frac{n(s,t)}{\bar{n}(s)} ds\Big)$$

$$> 0 \qquad \text{by Jensen inequality}$$

And a Poincaré inequality ?

$$\int_0^\infty \phi(s)\bar{n}(s)|u(s)|ds \leq \nu \left[\int b(s)\bar{n}(s)|u(s)|ds - \left|\int_0^\infty b(s)\bar{n}(s)u(s)ds\right|\right]$$

when

$$\int b(s)\bar{n}(s) = 1, \qquad \int u(s)\bar{n}(s)\phi(s)ds = 0$$

Theorem

If $b(\cdot) \geq \nu\phi(s)$, then the Poincaré inequality holds true.

Proof

$$\left|\int_0^\infty b(s)\bar{n}(s)u(s)ds\right| = \left|\int_0^\infty [b(s) - \nu\phi(s)]\bar{n}(s)u(s)ds\right|$$

$$\geq \int_0^\infty [b(s) - \nu\phi(s)]\bar{n}(s)|u(s)|ds$$

And a Poincaré inequality ?

$$\nu \int_0^\infty \phi(s)\bar{n}(s)|u(s)|ds \leq \int b(s)\bar{n}(s)|u(s)|ds - \Big|\int_0^\infty b(s)\bar{n}(s)u(s)ds\Big|$$

when

$$\int b(s)\bar{n}(s) = 1, \qquad \int u(s)\bar{n}(s)\phi(s)ds = 0$$

$$\Big|\int_0^\infty b(s)\bar{n}(s)u(s)ds\Big| = \Big|\int_0^\infty [b(s) - \nu\phi(s)]\bar{n}(s)u(s)ds\Big|$$

$$\geq \int_0^\infty [b(s) - \nu\phi(s)]\bar{n}(s)|u(s)|ds$$

$$= \int_0^\infty b(s)\bar{n}(s)|u(s)|ds - \int_0^\infty \nu\phi(s)\bar{n}(s)|u(s)|ds$$

Floquet case

The GRE is robust

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + r(s,t)\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s,t)\ n(s,t) ds \end{cases}$$

Assume that $r(s,t),\ b(s,t)$ are time periodic

Floquet case

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda_F + r(s,t)]\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s,t)\ n(s,t) ds \end{cases}$$

$r(s,t),\ b(s,t)$ are time periodic

Floquet case

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda_F + r(s,t)]\; n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s,t)\; n(s,t) ds \end{cases}$$

$r(s,t),\; b(s,t)$ are time periodic

$$\begin{cases} \frac{\partial \bar{n}(s,t)}{\partial t} + \frac{\partial \bar{n}(s,t)}{\partial s} + [\lambda_F + r(s,t)]\; \bar{n}(s,t) = 0, \\ \bar{N}(t) := \bar{n}(s=0,t) = \int_0^{+\infty} b(s,t)\; \bar{n}(s,t) ds \end{cases}$$

$$\frac{\partial \bar{\phi}(s,t)}{\partial t} + \frac{\partial \bar{\phi}(s,t)}{\partial s} - [\lambda_F + r(s,t)]\; \bar{\phi}(s,t) = \phi(0,t) b(s,t),$$

with

$$\bar{n}(s,t) > 0,\; \bar{\phi}(s,t) > 0 \text{ time periodic}$$

Floquet case

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda_F + r(s,t)]\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s,t)\ n(s,t) ds \end{cases}$$

The Generalized Relative Entropy is

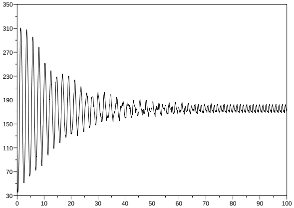
$$E(t) := \int_0^{\infty} \bar{\phi}(s,t)\bar{n}(s,t)H(\frac{n(s,t)}{\bar{n}(s,t)})ds$$

$$\frac{dE(t)}{dt} = -D_H(t)\phi(0,t)$$

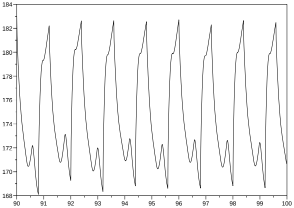
$$D_H(t) = \int b(s,t)\bar{n}(s,t)H\left(\frac{n(s,t)}{\bar{n}(s,t)}\right) ds - H\Big(\int b(s,t)\bar{n}(s,t)\frac{n(s,t)}{\bar{n}(s,t)} ds\Big)$$

$$> 0 \qquad \text{by Jensen inequality}$$

Floquet case



global simulation



zoom on the last periods

Total cell density $\int_0^\infty n(s,t)ds$ as a function of time

Elapsed time structured model

$$\frac{\partial n(s,t)}{\partial t} \quad \overbrace{+\frac{\partial n(s,t)}{\partial s}}^{\text{elapsed time advances}} \quad + \overbrace{r(s, bN(t))\ n(s,t)}^{\text{firing neurons}} = 0,$$

$$N(t) := n(s=0,t) = \underbrace{\int_0^{+\infty} r(s, bN(t))\ n(s,t) ds,}_{\text{reset of firing neurons}}$$

Theorem For small or large connectivity ($b > 0$ small or large) then desynchronization still holds

$$n(s,t) \underset{t\to\infty}{\longrightarrow} P_b(s)$$

Elapsed time structured model

$$\frac{\partial n(s,t)}{\partial t} + \overbrace{\frac{\partial n(s,t)}{\partial s}}^{\text{elapsed time advances}} + \overbrace{r(s,bN(t))\; n(s,t)}^{\text{firing neurons}} = 0,$$

$$N(t) := n(s=0,t) = \underbrace{\int_0^{+\infty} r(s,bN(t))\; n(s,t)ds,}_{\text{reset of firing neurons}}$$

Theorem For small or large connectivity ($b > 0$ small or large) then desynchronization still holds

$$n(s,t) \underset{t\to\infty}{\longrightarrow} P_b(s)$$

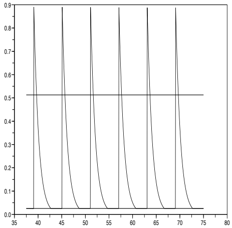
• In the middle range connectivity there are several periodic solutions (analytic forms of solutions),

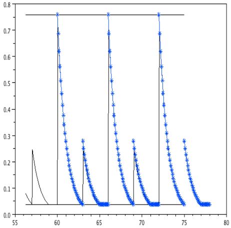
• These are stable (observed numerically).

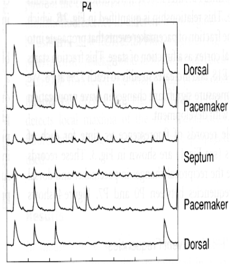
Elapsed time structured model

$$\frac{\partial n(s,t)}{\partial t} + \overbrace{\frac{\partial n(s,t)}{\partial s}}^{\text{elapsed time advances}} + \overbrace{r(s, bN(t))\; n(s,t)}^{\text{firing neurons}} = 0,$$

$$N(t) := n(s=0,t) = \underbrace{\int_0^{+\infty} r(s, bN(t))\; n(s,t)\,ds,}_{\text{reset of firing neurons}}$$

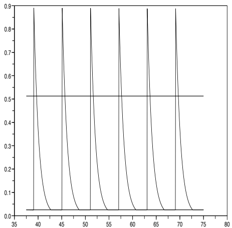


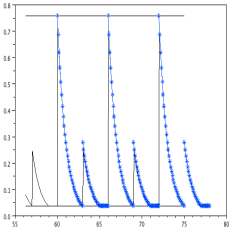


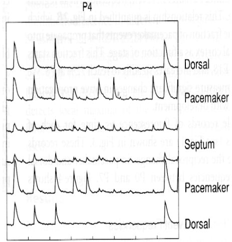


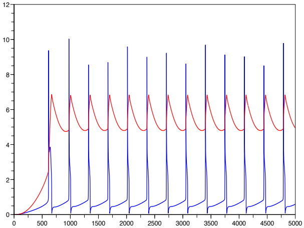
Right : Conhaim et al (2011) J. of physiology 589(10) 2529-2541.

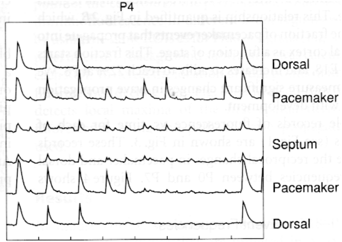
Comparison with I&F











Elapsed time structured model

Two equations

Time elapsed

$$\begin{cases} \partial_t n + \partial_s n + r(s)n = 0, \\ n(s=0,t) = \int_0^\infty r(s)n(s,t)ds =: N(t), \\ n(s,0) = n^0(s), \end{cases}$$

LIF

$$\begin{cases} \partial_t \widehat{n} + \partial_v[(-v+\sigma)\widehat{n}] - a\partial^2_{vv}\widehat{n} = N(t)\delta(v - V_R), \\ \widehat{n}(V_F,t) = 0, \qquad -a\partial_v \widehat{n}(V_F,t) = N(t). \end{cases}$$

Elapsed time structured model

From G. Dumont and J. Henry (linear equations)

Theorem There are functions $r(s)$, $q(v,s)$ such that, for solutions of

$$\begin{cases} \partial_t n + \partial_s n + r(s)n = 0, \\ n(s=0,t) = \int_0^\infty r(s)n(s,t)ds =: N(t), \\ n(s,0) = n^0(s), \end{cases}$$

then

$$\widehat{n}(v,t) = \int_0^\infty q(v,s)n(s,t)ds$$

satisfies, with the same $N(t)$,

$$\begin{cases} \partial_t \widehat{n} + \partial_v[(-v+\sigma)\widehat{n}] - a\partial^2_{vv}\widehat{n} = N(t)\delta(v-V_R), \\ \widehat{n}(V_F,t) = 0, \qquad -a\partial_v \widehat{n}(V_F,t) = N(t). \end{cases}$$

CONCLUSION

THANKS TO MY COLLABORATORS

M. J. Carceres

J. A. Carrillo

D. Smets

K. Pakdaman

D. Salort

C. Wainrib

CONCLUSION

THANK YOU FOR THE MARVELOUS ORGANIZATION

LISA

LUIGI

DIRK

FRANCO

MARCO

DARIO

THANK YOU ALL

CONCLUSION

THANKS TO MY COLLABORATORS

M. J. Carceres, J. A. Carrillo

D. Smets, D. Salort

K. Pakdaman

THANK YOU