PDEs for neural networks

analysis, simulations and behaviour

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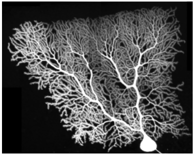
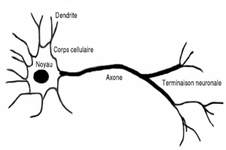
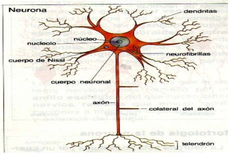
Centro Ennio De Giorgi, sept. 2017







Plan of the course

I. The single neuron,

I. 1. Excitable systems

I. 2. slow-fast dynamics,

I. 3. Integrate&Fire model, role of noise

II. Networks, I&F

III. Networks, time elapsed models

Single neuron

Electrically active cells are described by an **action potential** $V(t)$

Models are well established

- Hodgkin-Huxley
- FitzHugh-Nagumo
- Morris-Lecar
- Mitchell-Schaeffer

$$\begin{aligned}
C\frac{dv}{dt} &= I - g_{Na}m^3h(V - V_{Na}) - g_K n^4(V - V_K) - g_L(V - V_L)\\
\frac{dm}{dt} &= a_m(V)(1-m) - b_m(V)m\\
\frac{dh}{dt} &= a_h(V)(1-h) - b_h(V)h\\
\frac{dn}{dt} &= a_n(V)(1-n) - b_n(V)n\\
a_m(V) &= .1(V+40)/(1-\exp(-(V+40)/10))\\
b_m(V) &= 4\exp(-(V+65)/18)\\
a_h(V) &= .07\exp(-(V+65)/20)\\
b_h(V) &= 1/(1+\exp(-(V+35)/10))\\
a_n(V) &= .01(V+55)/(1-\exp(-(V+55)/10))\\
b_n(V) &= .125\exp(-(V+65)/80)
\end{aligned}$$

Single neuron

The class of Morris-Lecar is typically

$$\begin{cases} \frac{dV(t)}{dt} = \sum_{i=1}^{I} g_i(t)(V_i - V(t)) + I(t), \\ \dfrac{dg_i(t)}{dt} = \dfrac{G_i\big(V(t)\big) - g_i(t)}{\tau_i}, \qquad g_i(0) \geq 0, \quad i = 1, 2, ..., I, \end{cases}$$

- The index i refers to ionic channels along the nerve (Ca, K, Na, Cl...)
- The V_i are called the "reversal potentials" (Nernst-Planck theory)
- The leak V_L is used to aggregate some of them
- τ_i can be $\ll 1$
- Sharp nonlinearities G_i (sigmoids)

Single neuron

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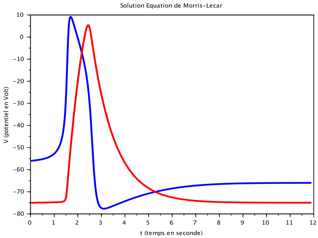
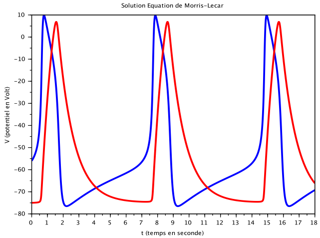
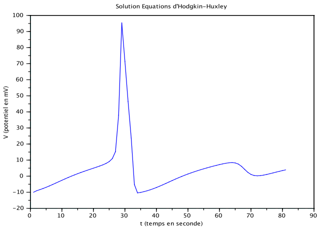
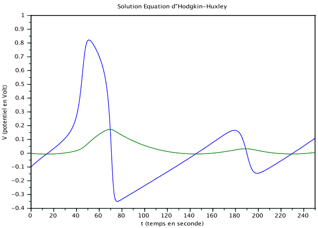


Hyperpolarization



Single neuron

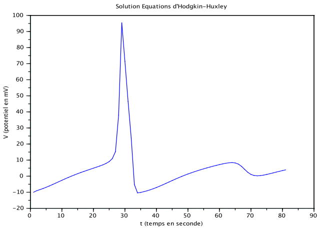


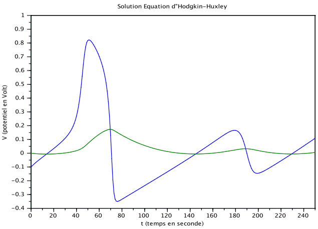


Solutions of Hodgkin-Huxley's model and of FitzHugh-Nagumo's model

- these models are accurate
- represent the property of **excitability** and **hyperpolarization**

Single neuron

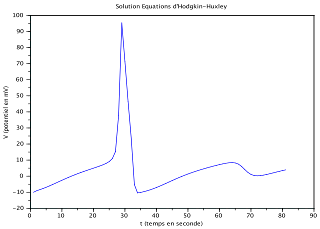


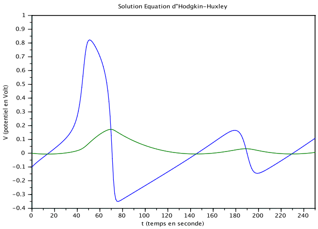


Solutions of Hodgkin-Huxley's model and of FitzHugh-Nagumo's model

- these models are accurate
- represent the property of **excitability**
  - A small perturbation generates a large trajectory
  - Return to equilibrium
  - The trajectory depends very little on the perturbation

Single neuron





Solutions of Hodgkin-Huxley's model and of FitzHugh-Nagumo's model

- These models are accurate BUT

- difficult to understand why they are **excitable**
- expensive for large assemblies of neurones
- do not explain properties of large assemblies
- This motivates using simpler models

Single neuron : FHN

FitzHugh-Nagumo

$$\begin{cases} \varepsilon \dot{v}(t) = f(v(t)) - w(t), & v(t=0) = v^0, \\ \dot{w}(t) = v(t) - v^* - \alpha w(t) & w(t=0) = w^0. \end{cases}$$

It can be derived from the Morris-Lecar model

$$\begin{cases} \frac{dV(t)}{dt} = g_L(V_L - V(t)) + G_{Na}(V(t))(V_{Na} - V(t)) + g_K(t)(V_K - V(t)) \\ \\ \dfrac{dg_K(t)}{dt} = \dfrac{G_K\big(V(t)\big) - g_K(t)}{\tau_K} \end{cases}$$

$$V_K < V_L < V_{Na}$$

Single neuron : FHN

FitzHugh-Nagumo

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$$v(t) = \ln(V(t) - V_K)$$

Single neuron : FHN

FitzHugh-Nagumo

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$$\frac{dv(t)}{dt} = \underbrace{\frac{g_L(V_L - V(t)) + G_{Na}(V(t))(V_{Na} - V(t))}{V(t) - V_K}}_{:=F(v(t))} - g_K(t)$$

Single neuron : FHN

FitzHugh-Nagumo

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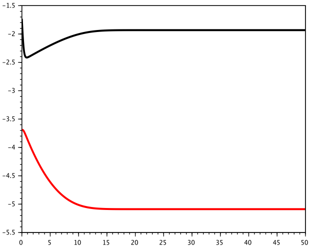

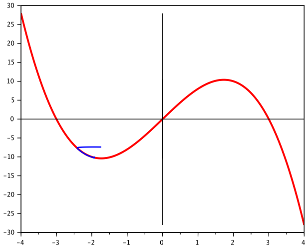

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Single neuron : FHN

FitzHugh-Nagumo

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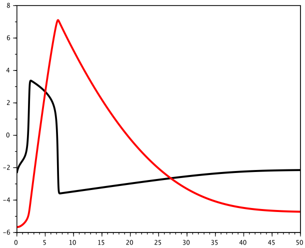

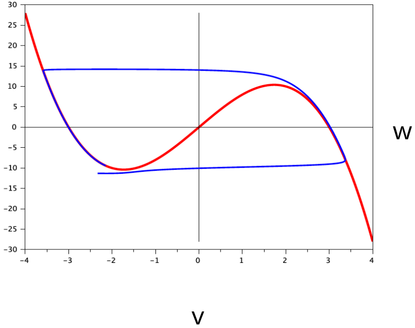

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FitzHugh-Nagumo

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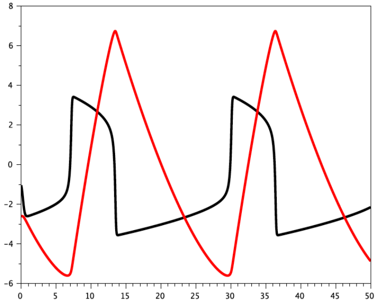

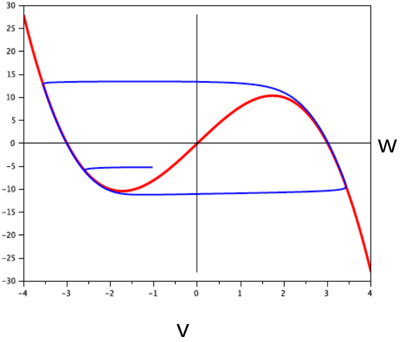

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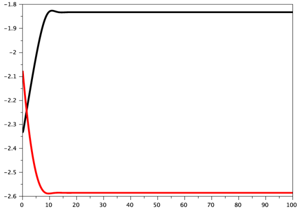
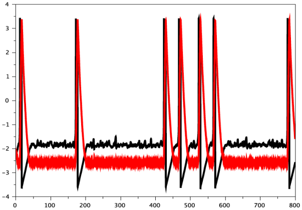
Single neuron : Role of noise





$$\begin{cases} \dfrac{dv(t)}{dt} = f(v(t)) - w(t), \\ \dfrac{dw(t)}{dt} = v(t) - v^* + \dfrac{dB(t)}{dt}. \end{cases}$$

Single neuron : S-F dynamics

Slow-fast dynamics

$$\begin{cases} \varepsilon \dot{v}_\varepsilon(t) = f(v_\varepsilon(t)) - w_\varepsilon(t), & v_\varepsilon(t=0) = v^0, \\ \dot{w}_\varepsilon(t) = v_\varepsilon(t) - v^* & w_\varepsilon(t=0) = w^0. \end{cases}$$

Theorem As $\varepsilon \to 0$, we have

- $v_\varepsilon(t) \to v$ a.e.,
- $w_\varepsilon(t) \to w(t)$ uniformly (locally)

and

$$\frac{dw(t)}{dt} = Q_\pm(w(t)) - v^*, \qquad v(t) = Q_\pm(w(t))$$

Single neuron : S-F dynamics

Slow-fast dynamics

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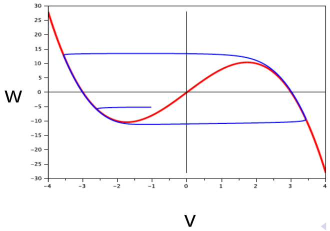
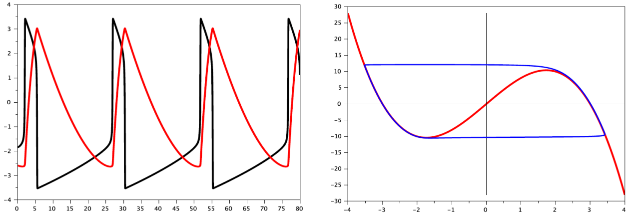
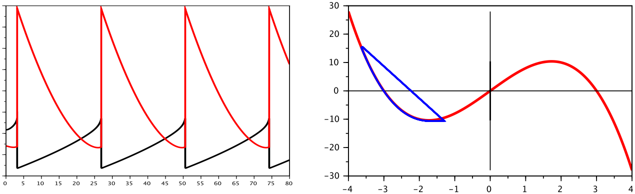


Single neuron : I&F



(FitzHugh-Nagumo, fast discharge) solution of a variant of the FitzHugh-Nagumo system

$$\begin{cases} \dfrac{dv(t)}{dt} = h(v(t)) + I(t), & \tau_i \leq t < \tau_{i+1}, \\ v(\tau_i) = V_R, & \lim_{t \to \tau_{i+1}} v(t) = v(\tau_{i+1}^-) = V_F. \end{cases}$$



Solution of the integrate-and-fire system

CONCLUSION

- Single neurone models are numerous and complex
- They share the property to describe excitability
- The I&F model is derived has a double Slow-Fast limit
- The I&F model simply describes the discharge at a determined threshold