Carl M. Bender: Nonlinear eigenvalue problems and PT symmetry

We discuss new kinds of nonlinear eigenvalue problems that are associated with instabilities, separatrix behavior, and hyperasymptotics. First, we consider the toy differential equation $y'(x) = \cos[\pi x y(x)]$, which arises in several physical contexts. We show that the initial condition $y(0)$ falls into discrete classes: $a_{n-1} < y(0) < a_n$ (n=1, 2, 3, ...). If $y(0)$ is in the nth class, $y(x)$ exhibits n oscillations. The boundaries an of these classes are analogous to quantum-mechanical eigenvalues and finding the large-n behavior of a_n is analogous to a semiclassical (WKB) approximation in quantum mechanics. For large n, $a_n \sim A\sqrt{n}$, where $A = 2^{5/6}$. The constant A is numerically close to the lower bound on the power-series constant P, which is fundamental in the theory of complex variables and which is associated with the asymptotic behavior of zeros of partial sums of Taylor series.

The Painlevé transcendents have a remarkable eigenvalue behavior. For example, as $n \to \infty$, the nth eigenvalue for P-I grows like $Bn^{3/5}$ and the nth eigenvalue for P-II grows like $Cn^{2/3}$. We calculate B and C analytically by reducing the Painlevé transcendents to linear eigenvalue problems in PT-symmetric quantum mechanics. We have also determined analytically the asymptotic behavior of the eigenvalues for P-IV.