Ancient solutions to Geometric Evolution Equations - Part I

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Introduction to Ancient and Eternal solutions

- We consider ancient and eternal solutions to geometric evolution equations such as: the Curve shortening flow the Ricci flow and the Yamabe flow.
- Definition: A solution to a parabolic equation is called ancient if it is defined for all time $-\infty < t < T$.

  If the solution is defined for all time $-\infty < t < +\infty$ it is called eternal.
- Ancient and eternal solutions appear as blow up limits near a singularity.
- An ancient solution is typically the blow up limit of a type I singularity, while an eternal solution is the blow up limit of a type II singularity.
- The classification of ancient and eternal solutions often plays a crucial role in understanding the singularities of the flow.

Outline of the talk and main results

- We will first discuss the classification of ancient solutions to the curve shortening flow and the Ricci flow on surfaces. This is joint work with R. Hamilton and N. Sesum.
- CSF: Let Γ_t be an ancient family of closed convex curves embedded in $\mathbb{R}^2$ which evolve by the curve shortening flow and exist for all time $-\infty < t < T$.

  We show that: Γ_t is either a family of contracting circles (type I) or a family of evolving Angenent ovals (type II).
- RF: Let $g(\cdot, t)$ be an ancient solution of the Ricci flow on a compact surface that exists for all time $-\infty < t < T$.

  We show that: $g(\cdot, t)$ is either a family of contracting spheres (type I) or one of the King-Rosenau solutions (type II).

The curve shortening flow

- Let Γ_t be a family of closed curves which is an embedded solution to the CSF, i.e. the embedding $F : \Gamma_t \to \mathbb{R}^2$ satisfies

$$\frac{\partial F}{\partial t} = -\kappa\,\nu$$

  with κ the curvature of the curve and ν the outer normal.

- *Gage and Hamilton:* if Γ_0 is convex, then the CSF shrinks Γ_t to a round point.

- *Grayson:* if Γ_0 is any embedded curve in $\mathbb{R}^2$, then the solution Γ_t to the CSF does not develop singularities before it becomes strictly convex.

- We assume from now on that Γ_t is an ancient convex solution to the CSF which defined on $I = (-\infty, 0)$ and shrinks to a point at $T = 0$.

The evolution of the curvature

- The curvature κ of Γ_t evolves, in terms of its arc-length s, by

$$\kappa_t = \kappa_{ss} + \kappa^3.$$

- If θ is the *angle between the tangent vector of* Γ_t *and the x-axis*, then on convex curves one can express κ as a function of θ and compute its evolution

$$\kappa_t = \kappa^2 \, \kappa_{\theta\theta} + \kappa^3.$$

- We introduce the pressure function $p = \kappa^2$ which evolves by

$$p_t = p \, p_{\theta\theta} - \frac{1}{2} \, p_\theta^2 + 2 \, p^2.$$

- We say that Γ_t is type I if

$$\sup_{\Gamma_t \times (-\infty, -1]} |t| \, p(\theta, t) < \infty.$$

Otherwise we say that Γ_t is of type II.

Examples and the Result

- Example of a type I solution (contracting circles):

$$p(\theta, t) = \frac{1}{2(-t)}, \quad t < 0.$$

- Example of a type II solution (Angenent ovals):

$$p(\theta, t) = \lambda\left(\frac{1}{1 - e^{2\lambda t}} - \sin^2(\theta + \gamma)\right), \quad t < 0$$

  with parameters $\lambda > 0$ and γ.
  As $t \to -\infty$ the Angenent ovals look like two grim reapers glued together.

- Theorem: The only ancient convex solutions to the CSF are the contracting spheres or the Angenent ovals.

- Open Question: Is the convexity assumption necessary ?

Sketch of proof - Monotone functional

- We will introduce a monotone functional $I(t)$ which depends on our solution and analyze its behavior as $t \to -\infty$ and as $t \to 0$ (vanishing time).
- Set $\alpha(\theta, t) := p_\theta(\theta, t)$. Then, α satisfies:

$$\alpha_t = p\,(\alpha_{\theta\theta} + 4\,\alpha).$$

- We introduce the functional

$$I(\alpha) = \int_0^{2\pi} (\alpha_\theta^2 - 4\alpha^2)\, d\theta.$$

- We have

$$\frac{d}{dt} I(\alpha(t)) = -2 \int_0^{2\pi} \frac{\alpha_t^2}{p}\, d\theta \le 0.$$

- In particular: $\lim_{t\to-\infty} I(\alpha)(t)$ exists or it is $+\infty$.

Sketch of proof - Classification of limits

- We show that

$$\lim_{t \to 0} I(\alpha(t)) = 0 \quad \text{and} \quad \lim_{t \to -\infty} I(\alpha(t)) = 0.$$

- Since

$$\frac{d}{dt} I(\alpha(t)) = -2 \int_0^{2\pi} \frac{\alpha_t^2}{p} \, d\theta \leq 0$$

we conclude that $I(\alpha(t)) \equiv 0$.

- This implies that $\alpha_t \equiv 0$. Hence $\alpha_{\theta\theta} + 4\alpha = 0$.
- Solving in θ gives: $\alpha := p_\theta = a(t) \cos 2\theta + b(t) \sin 2\theta$ and plugging back to the equation we conclude

$$p(\theta, t) = \frac{1}{-2t} \quad \text{or} \quad p(\theta, t) = \lambda \left(\frac{1}{1 - e^{2\lambda t}} - \sin^2(\theta + \gamma) \right).$$

Sketch of proof - $\lim_{t\to 0} I(\alpha(t)) = 0$

- Theorem (*Gage - Hamilton*) If Γ_0 is a closed convex curve embedded in the plane $\mathbb{R}^2$, the curve shortening flow shrinks Γ_t to a point in a circular manner. Moreover, the curvature $\tilde{\kappa}$ (and all the derivatives) of the rescaled flow converge to $\tilde{\kappa} = 1$ *exponentially*.
- Recalling that $\alpha := p_\theta = (\kappa^2)_\theta$ we show that

$$\lim_{t\to 0} I(t) = \lim_{t\to 0} \int_0^{2\pi} (\alpha_\theta^2 - 4\alpha^2)\, d\theta = 0$$

  by refining the above convergence result.
- Combining the above concluded the proof of our result.

Sketch of proof - $\lim_{t\to-\infty} I(\alpha(t)) \le 0$

- We recall that the *pressure* p satisfies $p_t = p\, p_{\theta\theta} - \frac{1}{2}\, p_\theta^2 + 2\, p^2$ and also that $p_t \ge 0$ (*Harnack estimate* for ancient solutions).
- A direct calculation shows that

$$\left(\frac{p_\theta^2}{2p}\right)_t \le p_\theta (p_\theta)_{\theta\theta} + 4p_\theta^2$$

- Integration by parts gives

$$\frac{d}{dt}\int_0^{2\pi} \frac{p_\theta^2}{2p}\, d\theta = \int_0^{2\pi} (-p_{\theta\theta}^2 + 4p_\theta^2)\, d\theta = -I(\alpha(t)).$$

- On the other hand, from the inequality $p_t \ge 0$ we obtain

$$\int_0^{2\pi} \frac{p_\theta^2}{2p}\, d\theta \le 2\int_0^{2\pi} p\, d\theta \le C$$

- Combining the above gives

$$\lim_{t\to-\infty} I(\alpha(t)) \le 0.$$

Conclusion

- We set $\alpha(\theta,t) := p_\theta(\theta,t)$ and showed that α satisfies:

$$\alpha_t = p\,(\alpha_{\theta\theta} + 4\,\alpha).$$

- We introduced the functional $I(\alpha) = \int_0^{2\pi} (\alpha_\theta^2 - 4\alpha^2)\, d\theta$ and showed that

$$\frac{d}{dt} I(\alpha(t)) = -2 \int_0^{2\pi} \frac{\alpha_t^2}{p}\, d\theta \le 0.$$

- We showed that

$$\lim_{t\to 0} I(\alpha(t)) = 0 \quad \text{and} \quad \lim_{t\to -\infty} I(\alpha(t)) = 0.$$

- Hence $\alpha_t \equiv 0$, implying $\alpha_{\theta\theta} + 4\alpha = 0$.
- Recalling that $\alpha = p_\theta$ and using the equation we conclude

$$p(\theta,t) = \frac{1}{-2t} \quad \text{or} \quad p(\theta,t) = \lambda\,\Big(\frac{1}{1-e^{2\lambda t}} - \sin^2(\theta+\gamma)\Big).$$

Ancient solutions of the Ricci flow on S^2

- Consider an ancient solution of the Ricci flow

$$\text{(RF)} \qquad \frac{\partial g_{ij}}{\partial t} = -2\, R_{ij}$$

  on S^2 that exists for all time $-\infty < t < T$ and becomes singular at time T.

- In dim 2, we have $R_{ij} = \frac{1}{2} R\, g_{ij}$, where R is the scalar curvature.

- *B. Chow, R. Hamilton:* After re-normalization, the metric becomes spherical at $t = T$.

- Choose a parametrization $g_{S^2} = d\psi^2 + \cos^2\psi\, d\theta^2$ of the limiting spherical metric and parametrize the (RF) by this, i.e. we write $g(\cdot, t) = u(\cdot, t)\, g_{S^2}$.

The equations for the conformal factor in different coordinates

- If $g_{ij} = u\, g_{S^2}$, then it was first observed by Angenent and L.F. Wu that the (RF) becomes equivalent to

$$u_t = \Delta_{S^2} \log u - 2, \quad \text{on } S^2 \times (-\infty, T).$$

- In Euclidean and Cylindrical coordinates where $g_{ij} = \bar{u}\, g_{euc} = \hat{u}\, g_{cyl}$ and

$$\hat{u}(s,\theta,t) = r^2\, \bar{u}(r,\theta,t), \qquad s = \log r$$

we have

$$\bar{u}_t = \Delta \log \bar{u}, \qquad \text{on } \mathbb{R}^2 \times (-\infty, T)$$

and

$$\hat{u}_t = \Delta_{cyl} \log \hat{u}, \qquad \text{on } \mathbb{R} \times [0, 2\pi] \times (-\infty, T)$$

The equation for the pressure

- We have just seen that if $g(\cdot, t) = u(\cdot, t)\, g_{S^2}$, then the (RF) becomes equivalent to:

$$u_t = \Delta_{S^2} \log u - 2, \quad \text{on } S^2 \times (-\infty, T).$$

- Assume from now on that $T = 0$.
- It is natural to consider the pressure function $v = u^{-1}$ which evolves by

$$\text{(PE)} \qquad v_t = v\, \Delta_{S^2} v - |\nabla v|^2 + 2v^2.$$

- Example of a type I solution (contracting spheres):

$$v(\psi, \theta, t) = \frac{1}{2(-t)}$$

The King-Rosenau solutions

- We look for explicit ancient solutions $g_{ij} = \bar{u}\, g_{euc}$ of the (RF) which become singular at time $T = 0$.
- J.R. King (1993) looked for radial solutions of equation (LFD) where the pressure function $\bar{v} := \bar{u}^{-1}$ is a polynomial function in $r := |x|$ with coefficients depending on t.
- A direct calculation shows that

$$\bar{v}(x,t) = a(t) + 2\, b(t)\, |x|^2 + a(t)\, |x|^4$$

where either $a(t) = b(t)$ (contracting spheres) or

$$a(t) = -\frac{\mu}{2} \operatorname{csch}(4\mu t), \quad b(t) = -\frac{\mu}{2} \coth(4\mu t), \quad \mu > 0.$$

- These solutions were independently discovered by P. Rosenau.
- The King-Rosenau solutions are not self-similar. As $t \to -\infty$ they look like two cigar (Barenblatt self similar) solutions glued together.

The classification result

Theorem: *(D., R. Hamilton, N. Sesum)*
An ancient solution to the (RF) on S^2 is either one of the contracting spheres or one of the King-Rosenau solutions.

Sketch of proof:

We recall that if $g(\cdot,t) = u(\cdot,t)\, g_{S^2}$ is the evolving metric, then the *pressure* $v = u^{-1}$ satisfies:

$$v_t = v\,\Delta_{S^2} v - |\nabla v|^2 + 2v^2 = R\,v > 0$$

- We show, by establishing sharp a'priori derivative estimates, that $v(\cdot,t) \xrightarrow{C^{1,\alpha}} v_\infty$, as $t \to -\infty$, for all $\alpha < 1$.
- Via a suitable Lyapunov functional we show that

$$R_\infty := \lim_{t\to-\infty} R(\cdot,t) = 0 \quad \text{a.e. on } S^2$$

and

$$v_\infty\,\Delta_{S^2} v_\infty - |\nabla v_\infty|^2 + 2v_\infty^2 = 0 \quad \text{a.e. on } S^2$$

Sketch of proof - Continuation

- We next classify the steady states v_∞ which satisfy

$$v_\infty \, \Delta_{S^2} v_\infty - |\nabla v_\infty|^2 + 2v_\infty^2 = 0.$$

Main Step: We show that v_∞ *has at most two zeros*.
We conclude that

$$v_\infty(\psi, \theta) = C \cos^2 \psi, \qquad \text{for } C \geq 0$$

namely that the pointwise limit as $t \to -\infty$ is a cylinder.

- If $C = 0$, then v must be a family of contracting spheres.
- If $C > 0$, then v must be one of the King-Rosenau solutions.

Lyapunov functional and convergence as $t \to -\infty$

- We recall that the *pressure* $v = u^{-1}$ satisfies:

$$\text{(PE)} \qquad v_t = v\,\Delta_{S^2} v - |\nabla v|^2 + 2v^2 = R\,v > 0.$$

Hence, the limit $v_\infty := \lim_{t\to-\infty} v(\cdot,t)$ exists and is bounded. Moreover, by a priori estimates:

$$v(\cdot,t) \xrightarrow{C^{1,\alpha}} v_\infty, \text{ as } t \to -\infty, \quad \forall \alpha < 1.$$

- We introduce the Lyapunov functional:

$$J(t) = \int_{S^n} \left(\frac{|\nabla v|^2}{v} - 4\,v \right) da$$

and show that $-C \le J(t) \le 0$ (for all $t \le t_0 < 0$) and

$$\frac{d}{dt} J(t) = -2 \int_{S^n} \frac{v_t^2}{v^2}\, da - 2 \int_{S^n} \frac{|\nabla v|^2}{v^2}\, v_t\, da < 0.$$

Lyapunov functional and convergence as $t \to -\infty$

- We have just seen that $-C \leq J(t) \leq 0$ and

$$\frac{d}{dt} J(t) = -2 \int_{S^n} \frac{v_t^2}{v^2}\, da - 2 \int_{S^n} \frac{|\nabla v|^2}{v^2}\, v_t \, da < 0.$$

- Integrating in time, yields (for any $\tau < 0$)

$$\int_{-\infty}^{\tau} \int_{S^n} \frac{v_t^2}{v^2}\, da + \int_{-\infty}^{\tau} \int_{S^n} \frac{|\nabla v|^2}{v^2}\, v_t \, da < \infty.$$

- Since and $v_t = Rv > 0$, it follows that

$$\liminf_{t \to -\infty} \int_{S^n} \frac{v_t^2}{v^2}(\cdot, t)\, da = 0.$$

- We conclude that the limit v_∞ is a $C^{1,\alpha}$ weak solution of the steady state equation $v_\infty \Delta_{S^2} v_\infty - |\nabla v_\infty|^2 + 2v_\infty^2 = 0.$

Classification of backward limits

- We have just seen that $v_\infty(\psi,\theta) := \lim_{t\to-\infty} v(\psi,\theta,t)$ is a $C^{1,\alpha}$ weak solution of the steady state equation

$$v_\infty \, \Delta_{S^2} v_\infty - |\nabla v_\infty|^2 + 2v_\infty^2 = 0.$$

- Theorem: There exists a conformal change of S^2 in which

$$v_\infty(\psi,\theta) := \lim_{t\to-\infty} v(\psi,\theta,t) = C \cos^2\psi, \quad C \geq 0$$

and the convergence is in $C^{1,\alpha}(S^2) \cap C^\infty(S^2 \backslash \{\text{poles}\})$.

- Main Step: Either $v_\infty \equiv 0$ or v_∞ has at most two zeros.

Sketch of the proof of the Main step

- Main Step: Either $v_\infty \equiv 0$ or v_∞ has at most two zeros.
- We express $g_{ij} = \bar{u}\, g_{euc}$ and recall that for each t, $\bar{u}(\cdot, t)$ is a solution of the elliptic equation

$$-\Delta \log \bar{u} = R\, \bar{u}, \qquad \text{on } \mathbb{R}^2.$$

- Recall that $R_\infty = 0$ a.e.
- Basic Lemma: Let $\delta > 0$ be a given small number. If for some $t \leq t_0$, $\rho < 1$ and $x_0 \in R^2$, with $|x_0| \leq r_0$, we have

$$\int_{B_\rho(x_0)} R\, \bar{u}\,(\cdot, t)\, dx \leq 4\pi - 2\delta$$

then: $\sup_{B_{\rho/4}(x_0)} \bar{u}(\cdot, t) \leq C(r_0, \rho, \delta)$.
- Since $\int_{\mathbb{R}^2} R\, \bar{u}(x,t)\, dx = 8\pi$, $\bar{u}_\infty := \lim_{t \to -\infty} \bar{u}(\cdot, t)$ is equal to infinity at most two points. Equivalently, $\bar{v}_\infty := \bar{u}_\infty^{-1}$ has at most two zeros.

The proof of the crucial L^∞ bound

- The proof of this L^∞ bound is inspired by a work of Brezis and Merle (1991). In particular we use the following inequality:
- Theorem (Brezis-Merle) Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain and let h be a solution of

$$\begin{cases} -\Delta h = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f \in L^1(\Omega)$. Then, for every $\delta \in (0, 4\pi)$ we have

$$\int_\Omega e^{\frac{(4\pi - \delta)|h(x)|}{\|f\|_{L^1(\Omega)}}} \, dx \le \frac{4\pi^2}{\delta} (\mathrm{diam}\, \Omega)^2.$$

- We apply the above inequality to $h := \log \bar{u}$, $f = -R\,\bar{u}$ and $\Omega = B_\rho(x_0)$ to show that if $\|f\|_{L^1(B_\rho(x_0))} < 4\pi - \delta$, then $\log \bar{u}$ is bounded in $B_{\rho/2}(x_0)$.

Classification of backward limits

- We will now conclude that

$$v_\infty(\psi,\theta) := \lim_{t\to-\infty} v(\psi,\theta,t) = C\,\cos^2\psi, \quad C \geq 0.$$

- Assume that v_∞ is not identically zero.
- Choose a conformal change of S^2 which brings the two possible zeros of v_∞ to two antipodal poles S, N on S^2.
- Perform Mercator's projection to map $S^2 \setminus \{S, N\}$ onto a cylinder $\mathcal{C}$ and set $\hat{v}(s,\theta,t) = v(\psi,\theta,t)\,\cosh^2 s$.
- We have

$$\lim_{t\to-\infty} \hat{v}(s,\theta,t) = \hat{v}_\infty(s,\theta) := v_\infty(\psi,\theta)\,\cosh^2 s > 0$$

uniformly on compact subsets of $\mathbb{R}\times[0,2\pi]$.

- The limit $\hat{v}_\infty$ satisfies $\Delta_{cyl}\,\hat{v}_\infty = 0$. We conclude that in our case $\hat{v}_\infty \equiv C$, for some $C \geq 0$, i.e. $v_\infty(\psi,\theta) = C\,\cos^2\psi$.

The King-Rosenau solutions

- Assuming that $v_\infty(\psi, \theta) = C \cos^2 \psi$, with $C > 0$, we will show that $v(\cdot, t)$ is one of the King-Rosenau solutions.
- We switch to plane coordinates, expressing

$$g = v^{-1} g_{S^2} = \bar{v}^{-1} (dx^2 + dy^2).$$

- To capture the King-Rosenau solutions we consider the scaling invariant quantity

$$Q(x, y, t) := \bar{v} \left[\left(\bar{v}_{xxx} - 3\bar{v}_{xyy}\right)^2 + \left(\bar{v}_{yyy} - 3\bar{v}_{xxy}\right)^2 \right].$$

- We observe that $Q \equiv 0$ on all three the King-Rosenau solutions, the cigar solutions and the cylinder solutions.
- We will show that $Q \equiv 0$ and conclude that $\bar{v}$ is one of the King-Rosenau solutions.

The King-Rosenau solutions

To establish that $Q \equiv 0$ we proceed as follows:

- We first show that $Q(t)$ is well defined on $\mathbb{R}^2$ and that

$$Q_{\max}(t) := \sup_{\mathbb{R}^2} Q(x,t)$$

  exists and is finite for all t.
- We then show that $Q_{\max}(t)$ is decreasing in t.
- We also show that $\lim_{t \to -\infty} Q_{\max}(t) = 0$.
- We conclude that $Q(\cdot, t) \equiv 0$, for all t.

The King-Rosenau solutions

- To establish that
$$\lim_{t\to-\infty} Q_{\max}(t) = 0$$
we use the fact that the only possible backward geometric limits are the cigar solutions or the cylinders.
- The proof involves a rather shuttle geometric argument by contradiction.
- To show that $Q_{\max}(t)$ is decreasing in t one considers the evolution equation of $Q(\cdot, t)$ and computes (after a long calculation done with mathematica !) that Q satisfies
$$Q_t - v\,\Delta Q \leq 0, \qquad \text{on } \mathbb{R}^2 \times (-\infty, 0).$$
Then you apply the maximum principle.

The case when the backward limit $v_\infty \equiv 0$

Proposition: If $v_\infty := \lim_{t\to-\infty} v(\cdot,t) \equiv 0$, then $v(\psi,\theta,t) = \frac{1}{2(-t)}$ (contracting spheres).

- Any simple closed curve γ with length $L(\gamma)$, divides the surface into two regions, with areas $A_1(\gamma)$ and $A_2(\gamma)$. We define the isoperimetric ratio

$$I(t) = \frac{1}{4\pi} \inf_\gamma L^2 \left(\frac{1}{A_1} + \frac{1}{A_2}\right) \leq 1.$$

- It is well known that $I \equiv 1$ iff the surface is a sphere.
- Hamilton computed that under the Ricci flow:

$$I'(t) \geq \frac{4\pi\,(A_1^2 + A_2^2)}{A_1 A_2 (A_1 + A_2)}\, I\,(1 - I^2).$$

- Since $A_1 + A_2 = 8\pi|t|$ and $A_1^2 + A_2^2 \geq 2A_1A_2$, we conclude the differential inequality

$$I'(t) \geq \frac{1}{|t|}\, I\,(1 - I^2).$$

The case when the backward limit $v_\infty \equiv 0$

- We argue by contradiction to show that $I(t_0) \equiv 1$.
- If $I(t_0) < 1$, for some $t_0 < 0$, it follows from the ODE that :

$$I(t) \le \frac{C_1}{|t|}, \qquad \forall t < t_0 < 0.$$

- We show that: $\exists t_k \downarrow -\infty$ and curves $\beta_k \in S^2$ s.t.

$$L_{S^2}(\beta_k) \ge c > 0 \quad \text{and} \quad L_{g(t_k)}(\beta_k) \le C < \infty.$$

- Since

$$L_{g(t_k)}(\beta_k) = \int_{\beta_k} \sqrt{u(t_k)}\, d_{S^2}$$

the above contradicts the fact that

$$\lim_{t \to -\infty} u(\cdot, t) = v_\infty^{-1} = +\infty.$$

Open problems - Mean Curvature flow

- Problem: Provide the classification of ancient convex compact solutions of the Mean Curvature flow in dimensions $n \geq 2$.
- The equivalent of the Angenent ovals in higher dimensions have been recently formally shown to exist by S. Angenent.
- The backward time limit, as $t \to -\infty$, of an n-dim Angenent oval is $\mathbb{R}^k \times S^{n-k}$.
- Xu-Jia Wang has classified the backward time limit of all ancient convex solutions to the n-dimensional MCF as S^n or $S^k \times R^{n-k}$ or the plane $\mathbb{R}^n$ of multiplicity two.
- Open Question: Are the contracting spheres and the Angenent ovals the only ancient convex compact solutions of the n-dim MCF ?

Open problems - 3-dim Ricci flow

- Problem: Provide the classification of ancient compact solutions of the 3-dimensional Ricci flow.
- Type I: The only type I compact ancient solutions are the contracting spheres.
- Brendle, Huisken and Sinestrari have shown that ancient solutions to the 3-dim Ricci flow that satisfy a pinching curvature condition are of type I.
- The analogue to the King-Rosenau solutions in dimension $n = 3$ have been shown to exist by Perelman.
- Other compact solutions in closed form have been found by V.A. Fateev in a paper dated back to 1996. These solutions are k collapsing.

Open problems - 3-dim Ricci flow

- The Perelman solutions are rotationally symmetric and type II. They are also k non-collapsing.
- The Perelman solutions are not given in closed form.
- The formal asymptotic behavior of the Perelman solutions as $t \to -\infty$ follows from a recent work of Angenent, Caputo and Knopf.
- Open Problem 1: Are all compact k non-collapsing solutions of the 3-dim Ricci flow radially symmetric ?
- Open Problem 2: Are the only radially symmetric solutions of the 3-dim Ricci flow the Perelman solutions ?