Singular Diffusion
The Ricci flow on Surfaces
Existence Uniqueness and Singularities

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Outline

- We will begin by a short introduction to fast diffusion eq
$$(*) \quad u_t = \Delta u^m, \qquad \text{on } \mathbb{R}^n \times (0, T)$$
in the range of exponents $0 \leq m < 1$.
- We will briefly discuss the Sobolev critical case $m = \frac{n-2}{n+2}$ in dimensions $n \geq 2$ and its connection to geometry.
- We will discuss the Cauchy problem for the limiting case
$$(**) \quad u_t = \Delta \log u, \qquad \text{on } \mathbb{R}^2 \times (0, T)$$
which corresponds to the Ricci flow on surfaces.
- Emphasis will be given to the singularity formation of the Ricci flow on $\mathbb{R}^2$.
- In particular we will discuss the vanishing behavior of solutions u of equation $(**)$ which define non compact metrics with finite area and the related problem of the classification of eternal solutions.

Fast Diffusion Equations

- Consider the non-linear of fast diffusion equation

$$u_t = \Delta u^m = \operatorname{div}\,(m\,u^{m-1}\,\nabla u), \qquad m < 1.$$

- It appears in physical applications such as diffusion in plasma and thin liquid film dynamics among other.
- We will consider nonnegative weak solutions, i.e. continuous functions $u \geq 0$ which satisfy the equation in distributional sense.
- Since, the diffusivity $D(u) = m\,u^{m-1} \uparrow +\infty$, as $u \downarrow 0$ the eq becomes singular at $u = 0$, resulting to fast-diffusion.
- This equation has been extensively studied and we will only briefly mention results that are relevant to the rest of our discussion.

The Aronson-Bénilan inequality

- If u is a solution to the fast-diffusion equation $u_t = \Delta u^m$, $m < 1$, then the pressure $v := \frac{m}{1-m}\, u^{-(1-m)}$ evolves by:

$$v_t = (1-m)\, v\, \Delta v - |\nabla v|^2.$$

- In the range of exponents $\frac{(n-2)_+}{n} < m < 1$, the pressure v satisfies the sharp Aronson-Bénilan inequality

$$(*_1) \qquad \Delta v \le \frac{\lambda}{t}, \qquad \lambda = \lambda(m,n) > 0.$$

which implies the following Li-Yau type differential inequality

$$(*_2) \qquad -v_t + (1-m)\,\lambda\,\frac{v}{t} \ge |\nabla v|^2.$$

- The differential inequality $(*_2)$ becomes an equality when v is the self-similar solution $U(x,t) = t^{-\lambda}\left(C + k\,\frac{|x|^2}{t^{2\mu}}\right)^{-\frac{1}{1-m}}$.

The Harnack Inequality

- Integrating the inequality $(*_2)$ on optimal paths gives the Harnack Inequality due to Auchmuty-Bao and Hamilton:

$$v(x_2,t_2) \le \left(\frac{t_2}{t_1}\right)^{\mu} \left[v(x_1,t_1) + \frac{\delta}{4} \frac{|x_2 - x_1|^2}{t_2^{\delta} - t_1^{\delta}}\, t_1^{\mu} \right]$$

  for $0 < t_1 < t_2$, with $\mu = \mu(m,n) > 0$ and $\delta = \delta(m,n) > 0$.
- Application: Solutions of $u_t = \Delta u^m$, with $\frac{(n-2)_+}{n} < m < 1$ satisfy the lower bound $u(x,t) \ge c(t)\,(1 + |x|^2)^{-\frac{1}{1-m}}$.
- Conclusion: Solutions become instantly strictly positive and remain so for all time.
- Remark: This is not true in the sub-critical range of exponents $m < \frac{(n-2)_+}{n}$, where solutions may vanish in finite time.

The other Aronson-Bénilan inequality

- A simple scaling argument shows that every solution u to the fast-diffusion equation $u_t = \Delta u^m$, $0 \leq m < 1$ satisfies the differential inequality

$$(*_3) \qquad u_t \leq \frac{u}{(1-m)\,t}.$$

- Integrating $(*_3)$ in time implies: $u(x,t_2) \leq u(x,t_1)\left(\frac{t_2}{t_1}\right)^{\frac{1}{1-m}}$, $\forall x \in \mathbb{R}^n$ i.e. the L^∞ norm of a solution doesn't blow up, if it is initially finite.
- Remark: In the range of exponents $\frac{(n-2)_+}{n} < m < 1$, solutions u exhibit a regularizing effect from $L^1_{\rm loc}$ to $L^\infty_{\rm loc}$:

$$\sup_{|x|\leq R} u(x,t) \leq F\left(t, R, \int_{B_{2R}} u_0(x)\,dx\right).$$

- This is not true when $m < \frac{(n-2)_+}{n}$.

The Cauchy problem in the super-critical case

Consider the fast-diffusion equation

$$(*) \qquad u_t = \Delta u^m, \qquad \frac{(n-2)_+}{n} < m < 1.$$

In the super-critical case no growth conditions need to be imposed on the initial data for existence. More precisely, it follows from the results of Herrero and Pierre and Dahlberg and Kenig:

- For any nonnegative continuous weak solution u of $(*)$, there exists a unique locally finite Borel measure μ_0 on $\mathbb{R}^n$ such that

$$\lim_{t \downarrow 0} u(\cdot, t) = \mu_0 \qquad \text{in} \quad D'(\mathbb{R}^n).$$

- The trace μ_0 determines the solution uniquely.
- For any locally finite Borel measure μ_0 on $\mathbb{R}^n$ there exists a continuous weak solution u of $(*)$ in $S_\infty = \mathbb{R}^n \times (0, \infty)$ with initial trace μ_0.

The sub-critical case $m < (n-2)_+/n$

- In the sub-critical case $m < \frac{(n-2)_+}{n}$ the analogues of the above results do not hold true. In particular, there exists no solution with initial data the Dirac mass.
- This makes the problem of the existence of solutions with initial data a measure a very delicate one.
- Optimal results in this directions have been obtained by E. Chasseigne and J.L. Vazquez, where a new class of weak solutions was introduced.
- The Sobolev critical case of exponents $m = \frac{n-2}{n+2}$ is of particular geometric interest as it corresponds to the Ricci flow for $n = 2$ and the Yamabe flow for $n \geq 3$.

The Ricci flow on $\mathbb{R}^2$

- In 1982 R. Hamilton introduced the Ricci flow, namely the evolution of a Riemannian metric g_{ij} by

$$\text{(RF)} \qquad \frac{\partial g_{ij}}{\partial t} = -2\, R_{ij}$$

where R_{ij} denotes the Ricci curvature of the metric g_{ij}.

- If $g_{ij} = u\, g_{euc}$, where g_{euc} denotes the standard Euclidean metric, then in dimension $n = 2$, we have

$$R_{ij} = \frac{1}{2}\, R\, g_{ij}, \qquad R = -\frac{\Delta \log u}{u}.$$

where R denotes the Scalar curvature of g_{ij}.

- Hence, in dim $n = 2$ the evolution of the metric $g_{ij} = u\, g_{euc}$ under the Ricci flow (RF) is equivalent to the equation:

$$\text{(LFD)} \qquad u_t = \Delta \log u.$$

- We will see how this equivalence is used to obtain geometric estimates for solutions of equation (LFD).

The Yamabe flow on $\mathbb{R}^n$, $n \geq 3$

- In 1987 R. Hamilton introduced the Yamabe flow, namely the evolution of a Riemannian metric $g_{ij} = v^{\frac{4}{n-2}} g_{euc}$ which is conformally equivalent to the standard Euclidean metric g_{euc} by

$$\text{(YF)} \qquad \frac{\partial g_{ij}}{\partial t} = -R\, g_{ij}$$

where R denotes the Scalar curvature of the metric g.

- Since, the scalar curvature R is given in terms of v by $R = -C_n\, v^{-\frac{n+2}{n-2}} \Delta v$, the function v satisfies the equation

$$(v^{\frac{n+2}{n-2}})_t = \Delta v$$

hence $u := v^{\frac{n+2}{n-2}}$ evolves by the fast-diffusion equation

$$u_t = \Delta u^{\frac{n-2}{n+2}}.$$

- The Yamabe flow was used to obtain a different proof of the Yamabe conjecture.

Logaritmic fast-diffusion

Consider the logarithmic fast-diffusion equation

$$(*) \quad u_t = \Delta \log u, \quad \text{in } \mathbb{R}^2 \times [0, T), \ \ T > 0.$$

- Lions and Toscani: $(*)$ arises as a singular limit for finite velocity Boltzmann kinetic models.
- Kurtz: $(*)$ describes the limiting density distribution of two gases moving against each other and obeying the Boltzmann equation.
- In dimension $n = 2$ equation $(*)$ arises as a model for long Va-der-Wals interactions in thin films of a fluid spreading on a solid surface, if certain nonlinear fourth order effects are neglected.

Examples of solutions of $u_t = \Delta \log u$ on $\mathbb{R}^2$

- Contracting spheres: $u(x,t) = \frac{8\,\lambda\,(T-t)}{(\lambda+|x|^2)^2}, \ \lambda > 0.$
  They are *ancient solutions* which vanish at time $t = T$ and:

$$\frac{d}{dt}\int_{\mathbb{R}^2} u\,dx = \int_{\mathbb{R}^2} R\,u = -4\pi.$$

- Cigar solution: $u(x,t) = \frac{1}{\lambda\,|x|^2 + e^{4\lambda t}}$.
  They are *eternal complete non-compact* solutions which look like cigars and have infinite area, i.e.

$$\int_{R^2} u\,dx = \infty.$$

- Cusp solution: $u(x,t) = \frac{2t}{|x|^2\,\log^2|x|}, \ |x| > 1.$
  They are *complete non-compact* solutions which look like cusps and have finite area.

Solutions on orbifolds

- Cylindrical change of coordinates: If $u(r, \theta, t)$ is a solution of

$$(*) \qquad u_t = \Delta \log u, \quad \text{on } \mathbb{R}^2 \times (0, T)$$

then

$$v(s, \theta, t) := r^2\, u(r, \theta, t), \quad s = \log r$$

is a solution of

$$(**) \qquad v_t = \Delta_c \log v, \quad \text{on } \mathbb{R} \times [0, 2\pi] \times (0, T)$$

where Δ_c denotes the cylindrical Laplacian.

- We seek for radial solutions $v^\mu(s, t)$ of $(**)$ with $T = 1$ in the self-similar form

$$v^\mu(s, t) = (1 - t)\, g(s - \gamma_\mu \log(1 - t)).$$

Solutions on orbifolds

- It follows that for each $\mu > 0$, there exists γ_μ and an one parameter family of solutions v^μ such that

$$\frac{d}{dt}\int_{-\infty}^{\infty} v^\mu(s,t)\,ds = -(2+\mu).$$

- Then, $u^\mu(x,t) := |x|^{-2}\, v(\log|x|, t)$ defines a radial solution of $u_t = \Delta \log u$ on $\mathbb{R}^2 \times (0,T)$ such that

$$\frac{d}{dt}\int_{\mathbb{R}^2} u^\mu(x,t)dx = -2\pi\,(2+\mu).$$

The u^μ, $\mu \neq 2$ define metrics on orbifolds. When $\mu = 2$ we recover the contracting spheres.

The Cauchy problem

Consider the Cauchy problem

$$(*)\quad \begin{cases} u_t = \Delta \log u & \text{in} \quad \mathbb{R}^2 \times [0, T) \\ u(\cdot, 0) = f & \text{on} \quad \mathbb{R}^2 \end{cases}$$

with initial data $f \geq 0$. In 1994, jointly with M. del Pino we obtained the following results:

- If $\int_{\mathbb{R}^2} f\, dx < \infty$, then $\forall \mu \geq 0$, $\exists\, u_\mu$ solution of $(*)$ on $\mathbb{R}^2 \times (0, T_\mu)$ with $T_\mu = \frac{1}{2\pi(2+\mu)} \int_{\mathbb{R}^2} f(x)\, dx$ satisfying

$$\frac{d}{dt} \int_{\mathbb{R}^2} u^\mu(x,t)\, dx = -2\pi\,(2+\mu).$$

- If $\int_{\mathbb{R}^2} f\, dx = \infty$, $\exists\, u$ solution of $(*)$ on $\mathbb{R}^2 \times (0, \infty)$.
- If $\int_{\mathbb{R}^2} f\, dx < \infty$, then every solution vanishes at time $T \leq T_{\max}$, with $T_{\max} = \frac{1}{4\pi} \int_{\mathbb{R}^2} f(x)\, dx$.

Remarks on the Cauchy problem

- The maximal solution ($\mu = 0$) defines complete non-compact metrics on $\mathbb{R}^2$ of finite area that behave as cusps at infinity.
- The intermediate solution u_μ with $\mu = 2$ corresponds to smooth metrics on S^2 evolving by the Ricci flow.
- All other solutions u_μ ($\mu \neq 0, 2$) correspond to metrics on orbifolds evolving by the Ricci flow.
- Estéban, Rodriguez and Vázquez : Radial u_μ, $\mu > 0$ are characterized by the outgoing flux at infinity:

$$\lim_{r\to\infty} r\,(\log u_\mu)_r = -(2+\mu).$$

- More generally, other solutions exist with non-constant flux at infinity:

$$\frac{d}{dt}\int_{\mathbb{R}^2} u^\mu(x,t)\,dx = -\varphi(t).$$

Vanishing behavior of solutions

If u is a solution of (*) with $\frac{d}{dt}\int_{\mathbb{R}^2} u(x,t)dx = -2\pi\,(2+\mu)$, then:

- $\mu = 2$ (Metrics on S^2)
  Y.S Hsu (also B. Chow, Hamilton):

$$u(x,t) \approx \frac{8\lambda\,(T-t)}{(\lambda + |x|^2)^2}, \quad \text{as}\, t \to T.$$

- $\mu > 2$, $0 < \mu < 2$ (Metrics on Orbifolds).
  Y.S. Hsu: Under radial symmetry, there exist unique constants $\alpha, \beta > 0$, $\alpha + 2\beta = 1$, depending on μ, and a parameter $\gamma > 0$ such that

$$u(x,t) \approx (T-t)^{\alpha}\,\phi_{\gamma}\Big(\frac{|x|}{(T-t)^{\beta}}\Big), \quad \text{as}\, t \to T.$$

where ϕ is a solution to the ODE

$$(r\phi'/\phi)'/r + \alpha\phi + \beta r\phi' = 0, \quad \phi_r(0) = 0, \phi(0) = \gamma.$$

The maximal solution

- Consider the maximal solution u of $u_t = \Delta \log u$. Its area decays as:

$$\frac{d}{dt}\int_{\mathbb{R}^2} u(x,t)\,dx = -4\,\pi.$$

  Hence u will vanish at time $T = \frac{1}{4\pi}\int_{\mathbb{R}^2} u_0(x)\,dx.$

- Estéban, Rodriguez and Vázquez : If u_0 is compactly supported, then

$$u(x,t) = \frac{2t}{|x|^2 \log^2 |x|}(1+o(1)), \qquad \forall t < T$$

  however the bound deteriorates as $t \to T$.

- It follows that for all $0 < t < T$, u defines a complete non-compact metric with finite area.

- Problem: Study the singularity formation of the metric $g := u\, g_{euc}$ as t approaches the vanishing time T.

Vanishing behavior of the maximal solution

- Jointly with M. del Pino and N. Sesum we established:

  i. On the outer region: $(T - t) \log |x| > T$, we have

$$u(x,t) \approx \frac{2T}{|x|^2 \log^2 |x|}, \quad \text{as } t \to T^-.$$

  ii. On the inner region: $(T - t) \log |x| < T$, u has the self-similar profile:

$$u(x,t) \approx (T-t)^2 \, e^{-\frac{2T}{T-t}} \, \phi(e^{-\frac{T}{T-t}} |x|)$$

  with $\phi(r) = \frac{2T^{-1}}{r^2+b}$ being the cigar metric.

- Our work is based on formal asymptotics previously derived by J.R. King in the rotationally symmetric case.

Our Geometric Estimates

- Our proof of the vanishing behavior of the maximal solution u is based on geometric estimates on the maximum curvature $R_{\max}$ and the width w of the evolving metric $g_{ij} := u\, g_{euc}$ near the vanishing time T of u.
- Definition of the width: Consider families $\mathcal{F}$ of curves Γ homotoping a circle at infinity to a point. Define the width of the metric $ds^2 = u\,(dx^2 + dy^2)$ on the plane

$$w = \inf_{F} \sup_{\Gamma \in \mathcal{F}} L(\Gamma)$$

where $L(\Gamma) = \int_\Gamma \sqrt{u}\, d\sigma$.

- Note: When $u = u(r)$ is rotationally symmetric then $w = \max_{0 \le r < \infty} 2\pi r \sqrt{u(r)}$.

Our Geometric Estimates

- Theorem [D., Hamilton] There exist constants $\gamma > 0$ and $C < \infty$ such that

$$\gamma\,(T-t) \le w \le C\,(T-t)$$

and

$$\frac{\gamma}{(T-t)^2} \le R_{\max} \le \frac{C}{(T-t)^2}$$

on $0 < t \le T$.

- The above estimates show that the singularity of the solution is of Type II. This is the first type II singularity which was shown to exists in the Ricci flow in any dimension.
- In the rotationally symmetric case $u = u(r,t)$:

$$\gamma\,(T-t) \le \max_{0 \le r < \infty} r\,\sqrt{u}(r,t) \le C\,(T-t)$$

Asymptotic Behavior as $t \to T$

- Assume that u_0 is compactly supported so that

$$u(x,t) = \frac{2t}{|x|^2 \log^2 |x|} \left(1 + o(1)\right), \quad \text{as } |x| \to \infty$$

for $t > 0$ and by the Aronson-Bénilan inequality $R \geq -1/t$.

Inner region convergence

- We first set $\bar{u}(x,\tau) = \tau^2\, u(x,t),\ \tau = \frac{1}{T-t}$.
- For $\tau_k \to \infty$ set

$$\bar{u}_k(y,\tau) = \alpha_k\, \bar{u}(\sqrt{\alpha_k}\, y, \tau + \tau_k)$$

where $\alpha_k = [\bar{u}(0,\tau_k)]^{-1}$ so that $\bar{u}_k(0,0) = 1$.

- It follows that $\bar{u}_k$ satisfies the equation

$$\bar{u}_\tau = \Delta \log \bar{u} + \frac{2\bar{u}}{\tau + \tau_k}, \quad -\tau_k + \frac{1}{T} < \tau < \infty.$$

- Set $\bar{R}_k = -\Delta \log \bar{u}_k / \bar{u}_k$. Our maximum curvature a-priori estimates imply that

$$-\frac{C}{(\tau+\tau_k)^2} \le \bar{R}_k(y,\tau) \le C.$$

Inner region convergence

- Theorem: Passing to a subsequence, $\bar{u}_k$ converges, as $\tau_k \to \infty$, uniformly on compact subsets to a complete eternal solution U of

$$U_\tau = \Delta \log U, \qquad \text{on } \mathbb{R}^2 \times (-\infty, +\infty)$$

of bounded width.

Eternal solutions

- The following classification result for eternal solutions of equation

$$(*) \quad u_t = \Delta \log u, \qquad \text{on } \mathbb{R}^2 \times (-\infty, \infty)$$

plays a crucial role in the proof of the inner convergence theorem. This is joint work with N. Sesum.

- Theorem: The only eternal solutions of (*) which are complete, satisfy the curvature bound $0 < R(\cdot, t) \leq C(t)$ and have bounded width, are the soliton (self-similar) solutions of the form

$$U(y, t) = \frac{1}{\lambda |y - y_0|^2 + e^{4\mu\tau}}, \quad \lambda, \mu > 0.$$

Outer scaling

- Outer Scaling: Expressing $u = u(r,\theta,t)$ in polar coordinates, we introduce the cylindrical coordinate change of variables:

$$v(\zeta,\theta,t) = r^2\, u(r,\theta,t), \quad \zeta = \log r$$

so that $v_t = \Delta_c \log v$.

- We perform a new scaling

$$\tilde{v}(\xi,\theta,\tau) = \tau^2 v(\tau\xi,\theta,t), \quad \tau = \frac{1}{T-t}.$$

Then $\tilde{v}$ satisfies the equation

$$\tau\,\tilde{v}_\tau = \frac{1}{\tau}\,(\log\tilde{v})_{\xi\xi} + \tau\,(\log\tilde{v})_{\theta\theta} + \tilde{v}_\xi + 2\tilde{v}.$$

- In addition: $\int \tilde{v}(\xi,\theta,\tau)\,d\theta d\xi = 2, \quad \forall\tau.$

Outer convergence

- Theorem (Outer Behavior): $\tilde{v}(\cdot,\tau)$ converges, as $\tau \to \infty$, to the $\theta-$independent solution V of the steady state equation

$$V_\xi + 2\,V = 0$$

given by

$$V(\xi) = \frac{2\,T}{\xi^2}, \text{ for } \xi > T, \quad V = 0, \text{ for } \xi < T.$$

Moreover, the convergence is uniform on the intervals $(-\infty, \xi^-]$ and $[\xi^+, +\infty)$, for all $-\infty < \xi^- < T < \xi^+ < +\infty$.

Conclusion

We conclude:

i. On the outer region: $(T-t)\log|x| > T$, we have

$$u(x,t) \approx \frac{2T}{|x|^2 \log^2|x|}, \quad \text{as } t \to T^-.$$

ii. On the inner region: $(T-t)\log|x| < T$, u has the self-similar profile:

$$u(x,t) \approx (T-t)^2 \, e^{-\frac{2T}{T-t}} \, \phi(e^{-\frac{T}{T-t}}|x|)$$

with $\phi(r) = \frac{2T^{-1}}{r^2+b}$ being the cigar metric.