Part 1
Introduction
Degenerate Diffusion and Free-boundaries

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Introduction

- We will discuss, in these lectures, certain geometric and analytical aspects of degenerate and singular diffusion.
- Models of degenerate diffusion include the Porous medium equation, the Gauss curvature flow and the Harmonic mean curvature flow.
- Models of singular diffusion include the Curve shortening flow, the Ricci flow on surfaces and the Yamabe flow.
- Emphasis will be given to the optimal regularity of solutions to degenerate diffusion and free-boundaries via sharp a priori estimates, to the analysis of solutions to geometric flows near singularities and to the classification of global solutions (ancient or eternal).

The Heat Equation

The simplest model of diffusion is the familiar heat equation:

$$u_t = \Delta u, \quad (x,t) \in \Omega \times [0,T], \quad \Omega \subset \mathbb{R}^n$$

(u is the density of heat, chemical concentration etc.)

Fundamental properties of the Heat equation:

- Smoothing Effect: Solutions become instantly *smooth*, at time $t > 0$.
- Infinite Speed of Propagation: Solutions with non-negative compactly supported initial data $u(\cdot, 0)$, become instantly *strictly positive*, at time $t > 0$.
- The Fundamental Solution:

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} \, e^{-\frac{|x|^2}{4t}}, \qquad t > 0.$$

A basic model of non-linear diffusion

We consider the simplest model of quasilinear diffusion:

$$(*) \qquad u_t = \Delta u^m = \operatorname{div}(m\, u^{m-1}\, \nabla u), \qquad u \geq 0$$

for various values exponents $m \in \mathbb{R}$.

- Porous medium equation $m > 1$:
  The diffusivity $D(u) = m\, u^{m-1} \downarrow 0$, as $u \downarrow 0$. $(*)$ becomes degenerate at $u = 0$, resulting to finite speed of propagation (Slow diffusion).
- Fast Diffusion $0 \leq m < 1$:
  The diffusivity $D(u) = m\, u^{m-1} \uparrow +\infty$, as $u \downarrow 0$. $(*)$ becomes singular at $u = 0$, resulting to Fast diffusion.
- Ultra-Fast Diffusion $m < 0$:
  When $m < 0$ we have ultra-fast diffusion with new interesting phenomena for example instant vanishing in some cases.
- Equation $(*)$ appears in many physical applications and in geometry (Ricci flow on surfaces and Yamabe flow).

Contraction of hyper-surfaces by functions of their principal curvatures

Consider the evolution of a hyper-surface Σ^n in $\mathbb{R}^{n+1}$ by the flow

$$\frac{\partial \mathbf{P}}{\partial t} = \sigma\, \mathbf{N}$$

where $\mathbf{P}$ is the position vector, $\mathbf{N}$ is a choice unit normal, and the speed σ is a smooth function of the principal curvatures λ_i of Σ.

Examples

- MCF: $\sigma = H = \lambda_1 + \cdots + \lambda_n$
- IMCF: $\sigma = -\frac{1}{H} = -\frac{1}{\lambda_1 + \cdots + \lambda_n}$
- GCF: $\sigma = K = \lambda_1 \cdots \lambda_n$
- GCF$^\alpha$: $\sigma = K^\alpha = (\lambda_1 \cdots \lambda_n)^\alpha$, $0 < \alpha < \infty$.
- HMCF: $\sigma = \frac{1}{\lambda_1^{-1} + \ldots + \lambda_n^{-1}}$.

Evolution Equations for Curvature flows

- CSF: Motion of a plane curve $y = u(x,t)$ by its Curvature

$$u_t = \frac{u_{xx}}{1+u_x^2}.$$

- MCF: Motion of a surface $z = u(x,y,t)$ in $\mathbb{R}^3$ by its Mean Curvature

$$u_t = \frac{(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy}}{1+|Du|^2}.$$

- GCF: Motion of a surface $y = u(x,y,t)$ in $\mathbb{R}^3$ by its Gaussian Curvature

$$u_t = \frac{\det D^2 u}{(1+|Du|^2)^{3/2}}.$$

It resembles the evolution Monge-Ampére equation.

Evolution Equations for Curvature flows

- HMCF: Motion of a surface $y = u(x, y, t)$ in $\mathbb{R}^3$ by its Harmonic Mean Curvature

$$u_t = \frac{\det D^2 u}{(1 + u_y^2) u_{xx} - 2 u_x u_y u_{xy} + (1 + u_x^2) u_{yy}}.$$

Remarks:

- The CSF and MCF are strictly parabolic and quasi-linear.
- The GCF and HMCF are fully-nonlinear.
- The GCF and HMCF are only weakly parabolic. They become degenerate (slow-diffusion) when the Gauss curvature $K = 0$.
- The HMCF becomes singular (fast-diffusion) as the mean curvature $H \to 0$.

The Questions - Motivation

These flows provide interesting models of non-linear diffusion where the interplay between analytical tools and geometric intuition lead to the development of new powerful techniques.

Many problems in image analysis use geometric flows: an image, represented by a gray-scale density function u, can be processed to remove noise by smoothing the level sets of u by a geometric flow.

Typical Questions

- Short and long time existence and regularity
- Free-boundaries
- Formation of singularities
- Classification of entire solutions (ancient or eternal)
- Existence through the singularities
- Final shape of the hyper-surface

Outline of lectures

- Part 1: Degenerate Diffusion and Free-boundary Regularity

  We will discuss the optimal regularity of solutions to different models of degenerate diffusion equations.
  After a brief discussion of the free-boundary regularity to the porous medium equation, we will discuss the regularity of weakly convex solutions to the Gauss curvature flow and the Harmonic mean curvature flow.
  We will also discuss the free-boundary regularity.

- Part 2: Singular Diffusion and Ancient solutions

  After a brief introduction to singular diffusion, we will discuss the existence, uniqueness and singularity formation in the Ricci flow on surfaces.
  The last two lectures will be devoted on new results on ancient solutions of the the curve shortening flow, the Ricci flow of surfaces, the Yamabe flow.

The Porous Medium Equation

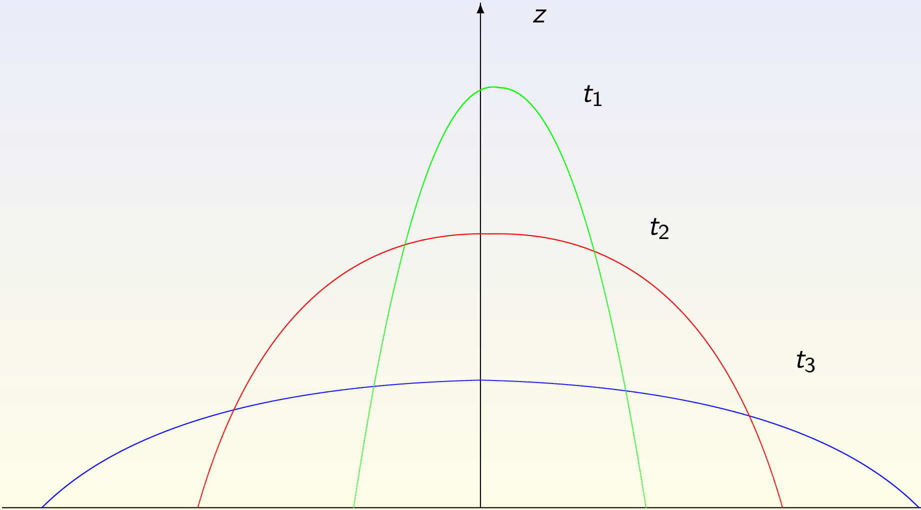
The simplest model of non-linear degenerate diffusion is the porous medium equation:

$$u_t = \Delta u^m = \operatorname{div}(m\, u^{m-1} \nabla u), \qquad m > 1.$$

- It describes various diffusion processes, for example the flow of gas through a porous medium, where u is the density of the gas and $v := u^{m-1}$ is the pressure of the gas.
- Since, the diffusivity $D(u) = m\, u^{m-1} \downarrow 0$, as $u \downarrow 0$ the equation becomes degenerate at $u = 0$, resulting to the phenomenon of finite speed of propagation.
- The Barenblatt solution: $U(x,t) = t^{-\lambda} \left(C - k \frac{|x|^2}{t^{2\mu}} \right)_+^{\frac{1}{m-1}}$ with $\lambda, \mu, k > 0$. It plays the role of the fundamental solution.

The Barenblatt Solution

$0 < t_1 < t_2 < t_3$



Finite Speed of propagation

The Barenblatt solution shows that solutions to the p.m.e have the following properties:

- Finite speed of propagation: If the initial data u_0 has compact support, then the solution $u(\cdot, t)$ will have compact support at all times t.
- Free-boundaries: The interface $\Gamma = \partial(\overline{\text{supp} u})$ behaves like a free-boundary propagating with finite speed.
- Solutions are not smooth: Solutions with compact support are only of class C^α near the interface.
- Weak solutions: We say that $u \geq 0$ is a weak solution of the equation $u_t = \Delta u^m$ in $Q_T := \Omega \times (0, T)$, if it is continuous on Q_T and satisfies the equations in the distributional sense.

Scaling

- Because it is nonlinear, the equation

$$(*) \qquad u_t = \Delta u^m, \qquad m \neq 1$$

has rich scaling properties.

- If u is a solution of $(*)$, then

$$v(x,t) := \frac{u(\alpha x, \beta t)}{\gamma}$$

is also a solution of (PM) if and only if

$$\gamma = \left(\frac{\alpha^2}{\beta}\right)^{1/(m-1)}.$$

The Aronson-Bénilan inequality

Aronson-Bénilan Inequality: Every solution u to the p.m.e. satisfies the differential inequality

$$(*) \qquad u_t \geq -\frac{k\,u}{t}, \qquad \lambda = \frac{1}{(m-1)+\frac{2}{n}}.$$

The *pressure* $v := \frac{m}{m-1}\,u^{m-1}$ which evolves by the equation

$$v_t = (m-1)\,v\,\Delta v + |\nabla v|^2$$

satisfies the sharp differential inequality

$$(**) \qquad \Delta v \geq -\frac{\lambda}{t}.$$

Remark: The Aronson-Bénilan $(*)$ inequality follows from $(**)$. The differential inequality $(**)$ becomes an equality when v is the Barenblatt solution.

The Li-Yau type Harnack inequality

- The Aronson-Bénilan inequality $\Delta v \geq -\frac{\lambda}{t}$ and the equation for v imply the Li-Yau type differential inequality:

$$v_t + (m-1)\,\lambda\,\frac{v}{t} \geq |\nabla v|^2.$$

- Integrating this inequality on optimal paths gives the following Harnack Inequality due to Auchmuty-Bao and Hamilton:

$$v(x_1, t_1) \leq \left(\frac{t_2}{t_1}\right)^{\mu} \left[v(x_2, t_2) + \frac{\delta}{4} \frac{|x_2 - x_1|^2}{t_2^{\delta} - t_1^{\delta}}\, t_2^{-\mu} \right]$$

if $0 < t_1 < t_2$, with $0 < \mu, \lambda < 1$ and $\delta > 0$.

- Application: If $v(0, T) < \infty$, then for all $0 < t < T - \epsilon$ we have:

$$v(x,t) \leq \; t^{-\mu}\,(T^{\mu}\,v(0,T) + C(n,m,\epsilon)\,|x|^2)$$

i.e. the pressure v grows at most quadratically as $|x| \to \infty$.

The Cauchy problem with general initial data

Let $u \geq 0$ be a weak solution of $u_t = \Delta u^m$ on $\mathbb{R}^n \times (0, T]$.

- The initial trace μ_0 exists; there exists a Borel measure μ such that

$$\lim_{t \downarrow 0} u(\cdot, t) = \mu_0 \qquad \text{in} \quad D'(\mathbb{R}^n)$$

  and satisfies the growth condition

$$(*) \qquad \sup_{R>1} \frac{1}{R^{n+2/(m-1)}} \int_{|x|<R} d\mu_0 < \infty.$$

- The trace μ_0 determines the solution uniquely.
- For every measure μ_0 on $\mathbb{R}^n$ satisfying $(*)$ there exists a continuous weak solution u of the p.m.e. with trace μ_0.
- All solutions satisfy the estimate $u(x, t) \leq C_t(u)\, |x|^{2/(m-1)}$, as $|x| \to \infty$.

The regularity of solutions

- Assume that u is a continuous weak solution of equation

$$u_t = \Delta u^m, \quad m > 1 \quad \text{on } Q := B_\rho(x_0) \times (t_1, t_2).$$

- Remark: It follows from parabolic regularity theory that if $u > 0$ in a parabolic domain Q, then $u \in C^\infty(Q)$.

  Proof: If $0 < \lambda \leq u \leq \Lambda$ in Q, then $u_t = \operatorname{div}(m\, u^{m-1} \nabla u)$ is strictly parabolic with bounded measurable coefficients.

  It follows from the *Krylov-Safonov* estimate that $u \in C^\gamma$, for some $\gamma > 0$, hence $D(u) := m\, u^{m-1} \in C^\alpha$.

  We conclude that from the *Schauder estimate* that $u \in C^{2+\alpha}$ and by repeating then same estimate we obtain that $u \in C^\infty$.

The regularity of solutions

- Question: What is the optimal regularity of the solution u ?
- Caffarelli and Friedman: The solution u is of class C^α, for some $\alpha > 0$.
- This result is, in some sense, optimal: The Barenblatt solution $U(x,t) = t^{-\lambda}\left(C - k\,\frac{|x|^2}{t^{2\mu}}\right)_+^{\frac{1}{m-1}}$ with $\lambda, \mu, k > 0$ is only of class C^α near the interface $u = 0$.
- Question: Is it true that $u^{m-1} \in C^{0,1}$?

The regularity of the free-boundary

- Assume that the initial data u_0 has compact support and let u be the unique solution of

$$u_t = \Delta u^m \quad \text{in } \mathbb{R}^n \times (0, \infty), \qquad u(\cdot, t) = u_0.$$

- Question: What is the optimal regularity of the free-boundary $\Gamma := \partial(\overline{\text{supp} u})$ and the solution u up to the free-boundary ?
- Caffarelli-Friedman: The free-boundary is always C^α.
- Caffarelli-Vazquez-Wolanski: The free-boundary is of class $C^{1+\alpha}$, for $t \geq t_0$ with t_0 sufficiently large.
- D.-Hamilton: Under certain non-degeneracy conditions the free-boundary is C^∞ smooth for $0 < t < \tau_0$.
- Koch: The free-boundary is of class C^∞, for $t \geq t_0$ with t_0 sufficiently large.

The equation for the pressure

Consider the Cauchy problem for the p.m.e:

$$\begin{cases} u_t = \Delta u^m & \text{in } \mathbb{R}^n \times (0,T) \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^n \end{cases}$$

with $u_0 \geq 0$ and compactly supported. It is more natural to consider the pressure $v = \frac{m}{m-1}\, u^{m-1}$ which satisfies

$$(*) \quad \begin{cases} v_t = (m-1)\, v\, \Delta v + |\nabla v|^2 & \text{in } \mathbb{R}^n \times (0,T) \\ v(\cdot,0) = v_0 & \text{in } \mathbb{R}^n. \end{cases}$$

Our goal is to prove the short time existence of a solution v of $(*)$ which is C^∞ smooth up to the interface $\Gamma = \partial(\overline{\text{supp}\, v})$. In particular, the free-boundary Γ will be smooth.

Short time C^∞ regularity

Non-degeneracy Condition: We will assume that the initial pressure v_0 satisfies:

$$(**) \qquad |\nabla v_0| \geq c_0 > 0, \qquad \text{at } \overline{\text{supp} v_0}$$

which implies that the free-boundary will start moving at $t > 0$.

Theorem (Short time Regularity) (D., Hamilton)
Assume that at $t = 0$, the pressure $v_0 \in C_s^{2+\alpha}$ and satisfies $(**)$. Then, there exists $\tau_0 > 0$ and a unique solution v of the Cauchy problem $(*)$ on $\mathbb{R}^n \times [0, \tau_0]$ which is smooth up to the interface Γ. In particular, the interface Γ is smooth.

Remark: The space $C_s^{2+\alpha}$ is Hölder space for second derivatives that it is scaled with respect to an appropriate singular metric s. This is necessary because of the degeneracy of our equation.

Short time Regularity - Sketch of proof for dimension $n = 2$

- To simplify the notation we assume that we are in dimension $n = 2$.
- Coordinate change: We perform a change of coordinates which fixes the free-boundary: Let $P_0 \in \Gamma(t)$ s.t.

$$v_x > 0 \ \text{ and } \ v_y = 0, \quad \text{at } P_0.$$

  Solve $z = v(x, y, t)$ near P_0 w.r to $x = h(z, y, t)$ to transform the free-boundary $v = 0$ into the fixed boundary $z = 0$.
- The function h evolves by the quasi-linear, degenerate equation

$$(\#) \quad h_t = (m-1)\, z \left(\frac{1+h_y^2}{h_z^2}\, h_{zz} - \frac{2h_y}{h_z}\, h_{zy} + h_{yy} \right) - \frac{1+h_y^2}{h_z}$$

- Outline: Construct a sufficiently smooth solution of $(\#)$ via the Inverse function Theorem between appropriate Hölder spaces, scaled according to a singular metric.

The Model Equation

- Our problem is modeled on the equation

$$h_t = z\,(h_{zz} + h_{yy}) + \nu\, h_z, \quad \text{on } z > 0$$

with $\nu > 0$.

- The diffusion is governed by the cycloidal metric

$$ds^2 = \frac{dz^2 + dy^2}{z}, \qquad \text{on } z > 0.$$

Its geodesics are cycloid curves.

- We define the distance function according to this metric:

$$\bar{s}((z_1, y_1), (z_2, y_2)) = \frac{|z_1 - z_2| + |y_1 - y_2|}{\sqrt{z_1} + \sqrt{z_2} + \sqrt{|y_1 - y_2|}}.$$

- The parabolic distance is defined as:

$$s((Q_1, t_1), (Q_2, t_2)) = \bar{s}(Q_1, Q_2) + \sqrt{|t_1 - t_2|}.$$

Hölder Spaces

- Let C_s^α denote the space of Hölder continuous functions h with respect to the parabolic distance function s.
- $$C_s^{2+\alpha}: \; h, h_t, h_z, h_y, z\,h_{zz}, z\,h_{zy}, z\,h_{yy} \in C_s^\alpha.$$
- Theorem (Schauder Estimate) Assume that h solves

$$h_t = z\,(h_{zz} + h_{yy}) + \nu\, h_z + g, \quad \text{on } Q_2$$

with $\nu > 0$ and $Q_r = \{0 \le z \le r, |y| \le r, t_0 - r \le t \le t_0\}$. Then,

$$\|h\|_{C_s^{2+\alpha}(Q_1)} \le C\,\left\{\|h\|_{C_s^0(Q_2)} + \|g\|_{C_s^\alpha(Q_2)}\right\}.$$

Proof: We prove the Schauder estimate using the method of approximation by polynomials introduced by L. Caffarelli and I. Wang.

Short time regularity - summary

- Using the Schauder estimate, we construct a sufficiently smooth solution of $(*)$ via the Inverse function Theorem between the Hölder spaces C_s^α and $C_s^{2+\alpha}$, which are scaled according to the singular metric s.
- Once we have a $C_s^{2+\alpha}$ solution we can show that the solution v is C^∞ smooth. Hence, the free-boundary $\Gamma \in C^\infty$.
- Remark: You actually need a global change of coordinates which transforms the free-boundary problem to a fixed boundary problem for a non-linear degenerate equation.

Long time regularity

- It is well known that the free-boundary will not remain smooth (in general) for all time. Advancing free-boundaries may hit each other creating singularities.
- Koch: (Long time regularity) Under certain natural initial conditions, the pressure v will be become smooth up to the interface for $t \geq T_0$, with T_0 sufficiently large.
- Question: Under what geometric conditions the interface will become smooth and remain so at all time ?

  Theorem (All time Regularity) (D., Hamilton and Lee)
  If the initial pressure v_0 is root concave, then the pressure v will be smooth and root-concave at all times $t > 0$. In particular, the interface will remain convex and smooth.

The Gauss Curvature Flow

Consider the deformation of a convex compact surface Σ^n in $\mathbb{R}^{n+1}$ by its Gauss curvature K:

$$\frac{\partial \mathbf{P}}{\partial t} = K \cdot \mathbf{N}, \qquad K = \lambda_1 \cdots \lambda_n.$$

- 1974 Firey: The GCF models the wearing process of tumbling stones subjected to collisions from all directions with uniform frequency. It shrinks strictly convex, *centrally symmetric* surfaces to round points.
- Tso: Existence and uniqueness for *strictly* convex and smooth intial data up to the time T when the surface Σ^n shrinks to a point.
- Andrews: *Firey's Conjecture*: the GCF shrinks strictly convex surfaces Σ^2 in $\mathbb{R}^3$ to spherical points.

The Regularity of solutions to GCF

If $z = u(x,t)$ defines Σ^n locally, then u evolves by the PDE

$$u_t = \frac{\det D^2 u}{(1+|Du|^2)^{\frac{n+1}{2}}}.$$

A strictly convex surface evolving by the GCF remains strictly convex and hence smooth up to its collapsing time T.

The problem of the regularity of solutions in the weakly convex case is a difficult question. It is related to the regularity of solutions of the evolution Monge-Ampére equation

$$u_t = \det D^2 u.$$

Question: What is the optimal regularity of weakly convex solutions to the Gauss Curvature flow ?

The Regularity of solutions to GCF -Known Results

- Hamilton: Convex surfaces with at most one vanishing principal curvature, will instantly become strictly convex and hence smooth.
- Chopp, Evans and Ishii: If Σ^n is $C^{3,1}$ at a point P_0 and two or more principal curvatures vanish at P_0, then P_0 will not move for some time $\tau > 0$.
- Andrews: A surface Σ^2 in $\mathbb{R}^3$ evolving by the GCF is always $C^{1,1}$ on $0 < t < T$ and smooth on $t_0 \leq t < T$, for some $t_0 > 0$. This is the optimal regularity in dimension $n = 2$.

  Remark: The regularity of solutions Σ^n in dimensions $n \geq 3$ poses a much harder question.
- Hamilton: If a surface Σ^2 in $\mathbb{R}^3$ has flat sides, then the flat sides will persist for some time.

Surfaces with Flat Sides

Consider a two-dimensional surface $\Sigma_t = (\Sigma_1)_t \cup (\Sigma_2)_t$ in $\mathbb{R}^3$ with $(\Sigma_1)_t$ flat and $(\Sigma_2)_t$ strictly convex.

- Hamilton: The flat side $(\Sigma_1)_t$ will shrink with *finite speed*.
- The curve $\Gamma_t = (\Sigma_1)_t \cap (\Sigma_2)_t$ behaves like a free-boundary propagating with finite speed. It will shrink to a point before the surface Σ_t does.
- Expressing the lower part of Σ_t as $z = u(x, y, t)$, we find the u evolves by

$$(*_1) \qquad u_t = \frac{\det D^2 u}{(1 + |Du|^2)^{3/2}}$$

with $u = 0$, at $(\Sigma_1)_t$ and $u > 0$, on $(\Sigma_2)_t$.

- The equation becomes degenerate at $u = 0$.

Rotationally Symmetric Examples

- If $z = u(r,t)$ is a rotationally symmetric solution of the (GCF), then u satifies by

$$u_t = \frac{u_r\, u_{rr}}{r\,(1+u_r^2)^{\frac{3}{2}}}.$$

- Model Equation (near $r \sim 1$):

$$u_t = u_r\, u_{rr}.$$

- Specific solutions of the model equation:

$$\phi_1 = (r + 2t - 1)_+^2$$

and

$$\phi_2 = \frac{(r-1)^3}{6\,(T-t)}.$$

- Conclusion: We expect that u vanishes quadratically at a moving the free-boundary.

The pressure function

- Since we expect quadratic behavior near $z = 0$ we introduce the pressure function $g = \sqrt{2u}$ and compute its evolution:

$$(*_2) \qquad g_t = \frac{g \, \det D^2 g + g_\nu^2 \, g_{\tau\tau}}{(1 + g^2 \, |Dg|^2)^{3/2}}.$$

- $(*_2)$ is a fully-nonlinear equation which becomes degenerate at the boundary of the flat side $\Gamma_t = (\Sigma_1)_t \cap (\Sigma_2)_t$.
- We assume that the initial surface satisfies the non-degeneracy condition

$$(\#) \quad g_\nu = |Dg| \geq c_0 > 0 \text{ and } g_{\tau\tau} \geq c_0 > 0, \text{ at } \Gamma_0.$$

with $\nu =$ inner normal to Γ_0 and $\tau =$ tangental to Γ_0.

GCF with flat sides -The Results

- Short time Regularity (D., Hamilton):
  If at $t = 0$, $g = \sqrt{2u} \in C_s^{2+\alpha}$ and satisfies $(\#)$, then there exists $\tau_0 > 0$ for which the solution g to the GCF is smooth (up to the interface Γ_t) on $0 < t \leq \tau_0$. In particular, the free-boundary Γ_t is smooth.

- Long time Regularity (D., Lee):
  The pressure g will remain smooth (up to the interface) up to the extinction time T_c of the flat side.

- Regularity at the Extinction time (D., Lee): At $t = T_c$ the surface Σ_t is at most of class $C^{2,\beta}$, $\beta < 1$.

Short time Existence - Outline of the proof

- Pressure Equation: We recall that the pressure p satisfies

$$(*_2) \qquad g_t = \frac{g \ \det D^2 g + g_\nu^2 \, g_{\tau\tau}}{(1 + g^2 \, |Dg|^2)^{3/2}}.$$

- Coordinate change: Let $P_0 \in \Gamma(t)$ s.t. $g_x > 0$ and $g_y = 0$ at P_0. Solve $z = g(x, y, t)$ near P_0 w.r to $x = h(z, y, t)$ to transform the f.b. $g = 0$ into the fixed boundary $z = 0$.
- h evolves by the fully-nonlinear degenerate equation:

$$(*_3) \qquad h_t = \frac{-z \ \det D^2 h + h_z h_{yy}}{(z^2 + h_z^2 + z^2 h_y^2)^{3/2}}, \quad z > 0.$$

- Outline: We construct a sufficiently smooth solution of $(*_3)$ via the Inverse Function Theorem between appropriate Hölder spaces, scaled according to a singular metric.

The Model Equation

- Our problem is modeled on the equation

$$h_t = z\, h_{zz} + h_{yy} + \nu\, h_z, \quad \text{on } z > 0$$

  with $\nu > 0$.
- The diffusion is governed by the singular metric

$$ds^2 = \frac{dz^2}{z} + dy^2, \qquad \text{on } z > 0$$

- We define the distance function according to this metric:

$$\bar{s}((z_1, y_1), (z_2, y_2)) = |\sqrt{z_1} - \sqrt{z_2}| + |y_1 - y_2|.$$

- The parabolic distance is defined as:

$$s((Q_1, t_1), (Q_2, t_2)) = \bar{s}(Q_1, Q_2) + \sqrt{|t_1 - t_2|}.$$

Hölder Spaces

- Let C_s^α denote the space of Hölder continuous functions h with respect to the parabolic distance function s.
- $$C_s^{2+\alpha}: \; h, h_t, h_z, h_y, z\, h_{zz}, \sqrt{z}\, h_{zy}, h_{yy} \in C_s^\alpha.$$
- Theorem (Schauder Estimate) Assume that h solves

$$h_t = z\, h_{zz} + h_{yy} + \nu\, h_z + g, \quad \text{on } Q_2$$

with $\nu > 0$ and $Q_r = \{0 \le z \le r^2, |y| \le r, t_0 - r^2 \le t \le t_0\}$. Then,

$$\|h\|_{C_s^{2+\alpha}(Q_1)} \le C\, \left\{ \|h\|_{C_s^0(Q_2)} + \|g\|_{C_s^\alpha(Q_2)} \right\}.$$

Proof: We prove the Schauder estimate using the method of approximation by polynomials introduced by L. Caffarelli and l. Wang.

Short time regularity - summary

- Using the Schauder estimate, we construct a sufficiently smooth solution of $(*)$ via the Inverse function Theorem between the Hölder spaces C_s^{α} and $C_s^{2+\alpha}$, which are scaled according to the singular metric s.
- Once we have a $C_s^{2+\alpha}$ solution we can show that the solution v is C^∞ smooth. Hence, the free-boundary $\Gamma \in C^\infty$.
- Observation: To obtain the optimal regularity, degenerate equations need to be scaled according to the right singular metric.
- Remark: You actually need a global change of coordinates which transforms the free-boundary problem to a fixed boundary problem for a non-linear degenerate equation.

Regularity up to the extinction T_c of the flat side

- Remark: By the result of Andrews the surface will become strictly convex before it shrinks to a point.
- Let us denote by T_c the extinction time of the flat side.
- The Result: The solution g will remain smooth up to the interface on $0 < t < T_c$.
- Main Step: The solution g is of class $C_s^{1+\alpha}$.
- Sketch of Proof: We show that $h \in C_s^{1+\alpha}$: each derivative $\tilde{h}$ of h satisfies a degenerate equation of the form

$$\tilde{h}_t = z a_{11} \tilde{h}_{zz} + 2\sqrt{z} a_{12} \tilde{h}_{zy} + a_{22} \tilde{h}_{yy} + b_1 \tilde{h}_z + b_2 \tilde{h}_y + H$$

with

$$a_{ij} = \begin{pmatrix} -h_{yy} & \sqrt{z}\, h_{zy} \\ \sqrt{z}\, h_{zy} & h_z - z\, h_{zz} \end{pmatrix}$$

and

$$b_1 = -\frac{h_{yy}}{h_z^3}.$$

I. A'priori Estimates

1. Ellipticity: $a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2 > 0, \;\; \forall \xi \neq 0.$
2. L^∞-Bound: $|a_{ij}| + |b_i| \leq \Lambda.$
3. Non-degeneracy: $b_1 \geq \nu > 0.$

Properties (1)-(3) follow from the next result which is shown by Pogorelov type computations.

- Theorem (D., Lee) Near $P_0 \in \Gamma_t$, where $g_x > 0$ and $g_y = 0$ we have:

$$0 < \lambda \leq \det a_{ij} \sim \frac{K}{g} \leq \frac{1}{\lambda}.$$

and

$$0 < \lambda \leq \operatorname{tr} a_{ij} \sim g_\nu^2\, g_{\tau\tau} + g\, g_{\nu\nu} \leq \frac{1}{\lambda}.$$

- Notation:
  $K =$ Gauss Curvature
  $\nu =$ direction normal to the level sets of g
  $\tau =$ direction tangential to the level sets of g.

II. Hölder Continuity of solutions to degenerate equations in non-divergence form

- Theorem (D., Lee) Let h be a classical solution of equation

$$h_t = z a_{11} h_{zz} + 2\sqrt{z} a_{12} h_{zy} + a_{22} h_{yy} + b_1 h_z + b_2 h_y + H$$

on Q_2. Assume that there exist constants λ and ν such that

$$a_{ij}\xi\,\xi_j \geq \lambda|\xi|^2,\ \forall \xi \in \mathbb{R}^2, \quad |a_{ij}| + |b| \leq \frac{1}{\lambda}, \quad \frac{b_1}{2a_{11}} \geq \nu.$$

Then, there exists $\alpha = \alpha(\lambda, \Lambda, \nu)$ such that

$$\|h\|_{C_s^\alpha(Q_1)} \leq C\, \left\{ \|h\|_{C^0(Q_2)} + \|H\|_{L^3_\sigma(Q_2)} \right\}$$

where $d\sigma = z^{\frac{\nu}{2}-1}\, dzdydt$ and

$$Q_r = \{0 \leq z \leq r^2, |y| \leq r, t_0 - r^2 \leq t \leq t_0\}.$$

- Remark: The above extends the Krylov-Safonov C^α-regularity result to the degenerate case.

Conclusion

Combining the a-priori estimates and the Hölder Regularity result, we conclude that:

- $h \in C_s^{1,\alpha} \Rightarrow g \in C_s^{1,\alpha}$.
- (Classical Regularity + Scaling) $\Rightarrow g \in C_s^{2+\alpha}$.
- (Local Estimates) $\Rightarrow$ g is C^∞-up to interface Γ up to the extinction time T_c of the flat side.

Further regularity results in dimensions $n \geq 3$

Question: What is the optimal regularity of solutions to the GCF in dimensions $n \geq 3$?

- Short time existence of solutions Σ_t^n to the GCF with flat sides such that $\Sigma_t^n \in C^{1,\frac{2}{n}}$ for $0 < t < \tau_0$.
- Long time regularity (Open Problem): Is a surface Σ_t^n with flat sides going to remain of class $C^{1,\frac{n}{2}}$ until it becomes strictly convex ?
- $C^{1,\alpha}$-Regularity : Are solutions of the GCF (or the evolution Monge-Ampére equation) always of class $C^{1,\alpha}$, for $\alpha > 0$?
- Final Shape (Open Problem): What is the final spape of the surface Σ_t^n as it shrinks to a point ? Does the surface become asymptotically spherical ?

Optimal regularity results in dimensions $n \geq 3$

Theorem : (D., Savin) Solutions to the GCF in dimension $n = 3$ are always of class $C^{1,\alpha}$.

Example: (D., Savin) In dimensions $n \geq 4$ there exist self-similar solutions of $u_t = \det D^2 u$ with edges persisting.

Theorem : (D., Savin) If the initial surface is of class $C^{1,\beta}$, then solutions Σ_t^n to the GCF for $n \geq 3$ are of class $C^{1,\alpha}$, for $0 < \alpha \leq \beta$.

Remark: Same results hold for motion by K^p, $p > 0$ and for viscosity solutions to

$$\lambda\,(\det D^2 u)^p \leq u_t \leq \Lambda\,(\det D^2 u)^p$$

for $0 < \lambda < \Lambda < \infty$ and $p > 0$.

Open Problem: Is a surface Σ_t^n with flat sides going to remain of class $C^{1,\frac{n}{2}}$ until it becomes strictly convex ?

The Q_k Curvature Flows

Consider a compact convex hypersurface Σ^n in $\mathbb{R}^{n+1}$ evolving by the Q_k-flow for $1 \leq k \leq n$

$$\frac{\partial \mathbf{P}}{\partial t} = Q_k \, \mathbf{N}$$

with speed $Q_k(\lambda) = \frac{S_k(\lambda)}{S_{k-1}(\lambda)} = \frac{\sum \lambda_{i_1} \cdots \lambda_{i_k}}{\sum \lambda_{i_1} \cdots \lambda_{i_{k-1}}}$.

- $Q_1 = H$ (Mean Curvature)
- $Q_n = \frac{1}{\lambda_1^{-1} \cdots \lambda_n^{-1}}$ (Harmonic Mean Curvature)
- Andrews: Existence of smooth solutions with strictly convex and smooth initial data up to the time T when the surface shrinks to a point. The surface becomes spherical as $t \to T$.
- Dieter: Convex surfaces with $S_{k-1} > 0$ become instantly smooth.
- Caputo, D., Sesum: *Long time existence* of $C^{1,1}$ convex solutions (not strictly convex).

Consider a compact surface Σ_t in $\mathbb{R}^3$ with flat sides evolving by the (HMCF)

$$\frac{\partial \mathbf{P}}{\partial t} = \kappa \cdot \mathbf{N}, \qquad \kappa(\lambda_1, \lambda_2) = \frac{\lambda_1 \, \lambda_2}{\lambda_1 + \lambda_2} = \frac{K}{H}.$$

The resulting PDE is fully-nonlinear, weakly parabolic but becomes degenerate at points where $K = 0$ and singular when $H \to 0$. In the latter case the flow is not defined.
The linearized operator $\mathcal{L}$ is given by

$$\mathcal{L} = a^{ik} \, \nabla_i \, \nabla_k u, \qquad a^{ik} = \frac{\partial}{\partial h^i_k} \left(\frac{G}{H} \right).$$

Notice that in geodesic coordinates around a point at which the second fundamental form matrix $A = \mathrm{diag}(\lambda_1, \lambda_2)$ we have

$$(a^{ik}) = \mathrm{diag}\left(\frac{\lambda_2^2}{(\lambda_1 + \lambda_2)^2}, \frac{\lambda_1^2}{(\lambda_1 + \lambda_2)^2}\right)$$

Highly degenerate convex HMCF

Let $\Sigma_t = (\Sigma_1)_t \cup (\Sigma_2)_t$ in $\mathbb{R}^3$ with $(\Sigma_1)_t$ flat and $(\Sigma_2)_t$ strictly convex. Let $\Gamma_t = (\Sigma_1)_t \cap (\Sigma_2)_t$ denote the interface.

Expressing the lower part of Σ as a graph $z = u(x, y, t)$, we find that u evolves by:

$$u_t = \frac{\det D^2 u}{(1+u_x^2)u_{yy} - 2\, u_x u_y u_{xy} + (1+u_y^2)u_{xx}}.$$

Theorem: (D., Caputo:) If Σ_0 is of class $C^{k,\alpha}$, $k \geq 1$, $0 < \alpha \leq 1$ then: the HMCF admits a viscosity solution of class $C^{k,\alpha}$ with pressure smooth up to the *interface* $\Gamma_t = (\Sigma_1)_t \cap (\Sigma_2)_t$.

The flat side persists for some positive time and the interface Γ_t is smooth and evolves by the Curve Shortening Flow.

The Q_k-flow in dim $n \geq 3$

In the case of a convex surface Σ^n in $\mathbb{R}^{n+1}$ with flat sides evolving by the Q_k-flow

$$\frac{\partial \mathbf{P}}{\partial t} = Q_k \cdot \mathbf{N}, \qquad Q_k = \frac{S_k(\lambda)}{S_{k-1}(\lambda)}$$

then a similar result was recently shown:

- Caputo, D., Sesum. *Short time existence* of a solution with flat sides which is smooth up to the interface. In particular, the interface moves by the $n-1$ dimensional Q_{k-1} flow.
- Caputo, D., Sesum: *Long time existence* of $C^{1,1}$ convex solutions (not strictly convex).

  Question: What is the optimal regularity of solutions ?

  Question: Do solutions become strictly convex before they shrink to a point ?

HMCF on star-shaped surfaces with $H > 0$

We assume that the initial surface Σ_0 is compact, star-shaped, of class $C^{2,1}$, and has strictly positive mean curvature $H > 0$.
Jointly with Natasa Sesum:

- Short time existence: There exists $\tau > 0$, for which the HMCF admits a $C^{2,1}$ solution Σ_t, such that $H > 0$ on $t \in [0, \tau)$.
- Long time existence: Let $T = A/4\pi$, with A the initial surface area. Then one of the following holds:

  (i) $H \to 0$ at some point $P_0 \in \Sigma_{t_0}$, at time $t_0 < T$, or

  (ii) a $C^{1,1}$ solution to the flow exists up to T, it becomes strictly convex at time $T_c < T$, and it shrinks to a point at time T.
- Question: Does $H \to 0$ before the time T ?
- On a surface of revolution with $H > 0$, we have $H > 0$ up to $T = A/4\pi$. Hence, Σ_t exists up to T and shrinks to a point at T. Moreover, $\Sigma_t \in C^\infty$, $0 < t < T$.
- Open problem: Higher dimensions.

Closing Remarks

- Degenerate equations with certain structure have smoothing properties once their solutions are scaled with respect to the appropriate singular metric.

  As a result, one may obtain the C^∞ -regularity in a number of degenerate free-boundary problems related to curvature flows.

- In other cases the solutions do not become regular and one needs to develop more sophisticated techniques to establish the optimal regularity of solutions or pass through their singularities.

  Open Problem: Find a flow which will take a non-convex surface with certain geometric properties and deform it to a smooth convex surface.