REACTION-DIFFUSION ELLIPTIC EQUATIONS: REGULARITY AND SYMMETRY OF MINIMIZERS

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Abstract. We will present recent developments on solutions of reaction-diffusion elliptic equations that are related to some classical results in the theory of minimal surfaces. The connection between reaction-diffusion elliptic equations and minimal surfaces originates in semilinear models of phase transitions. As the reaction term becomes stronger, interfaces between two states tend to minimize their area.

A classical result in minimal surface theory is the flatness of minimal graphs up to dimension 7. Its semilinear analogue is a conjecture posed by E. De Giorgi in 1978 for the Allen-Cahn equation $-\Delta u = u - u^3$, and for which progress has been made recently. We will describe these developments. They establish rich relations among different qualitative properties of solutions: their stability, minimality, monotonicity in one variable, and their one-dimensional symmetry. We will also study saddle-shaped solutions to the Allen-Cahn equation in all of $\mathbb{R}^{2m}$. They play an important role in the semilinear analogue of a classical result: the existence of singular minimal cones in high dimensions, an important example being the Simons cone $\mathcal{C} = \{(x^1, x^2) \in \mathbb{R}^m \times \mathbb{R}^m : |x^1| = |x^2|\}$. Saddle-shaped solutions are odd with respect to the Simons cone and depend only on the two radial variables $|x^1|$ and $|x^2|$. Thus, they solve a PDE in a quarter of the plane $\mathbb{R}^2$.

We will also present related results for semilinear equations posed in bounded domains, such as the explosion or Gelfand-type problem. As in minimal surfaces theory and as in the conjecture of De Giorgi, we have boundedness or regularity of stable solutions and local minimizers in low space dimensions —while singular ones exist in higher dimensions. We will present recent techniques leading to these results.

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