Hyperbolic Equations with non Lipschitz Coefficients

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(joint work with F. Colombini, F. Fanelli and G. Métivier)

Università di Trieste

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Suppose that L is *strictly hyperbolic* with bounded coefficients, i.e. there exist $\lambda_0, \Lambda_0 > 0$ such that

$$\lambda_0|\xi|^2 \leq \sum_{j,k=1}^{n} a_{jk}(t)\xi_k\xi_j \leq \Lambda_0|\xi|^2 \tag{2}$$

for all $t \in [0, T]$ and for all $\xi \in \mathbb{R}^n$.

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for all $t \in [0, T]$ and for all $\xi \in \mathbb{R}^n$.
It is well–known that if the coefficients a_{jk} are *Lipschitz–continuous* (it is sufficient absolutely continuous) then an *energy estimate* holds for L.

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In particular it is possible to prove that for all $s \in \mathbb{R}$ there exists $C_s > 0$ such that

$$\begin{aligned}\sup_{0\le t\le T} &\{\|u(t,\cdot)\|_{\mathcal{H}^{s+1}} + \|\partial_t u(t,\cdot)\|_{\mathcal{H}^s}\} \\ &\le C_s(\|u(0,\cdot)\|_{\mathcal{H}^{s+1}} + \|\partial_t u(0,\cdot)\|_{\mathcal{H}^s} + \int_0^T \|Lu(t,\cdot)\|_{\mathcal{H}^s}\, dt),\end{aligned} \tag{3}$$

for every function $u \in \mathcal{C}^0([0,T],\mathcal{H}^{s+1}(\mathbb{R}^n)) \cap \mathcal{C}^1([0,T],\mathcal{H}^s(\mathbb{R}^n))$ with $Lu \in \mathcal{L}^1([0,T],\mathcal{H}^s(\mathbb{R}^n))$, in particular for all $u \in \mathcal{C}^2([0,T],\mathcal{H}^\infty(\mathbb{R}^n))$.

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The estimate (3) implies that the *Cauchy problem*

$$\begin{cases} Lu = \partial_t^2 u - \displaystyle\sum_{j,k=1}^n a_{jk}(t)\partial_{x_j}\partial_{x_k} u = f & \text{in } (0,T) \times \mathbb{R}^n, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1 \quad \text{in } \mathbb{R}^n. \end{cases} \tag{4}$$

is well–posed in Sobolev spaces (*without loss of derivatives*).

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Suppose that there exists $C > 0$ such that

$$\int_0^{T-\varepsilon} |(a_{jk}(t+\varepsilon) - a_{jk}(t))|\, dt \leq C\varepsilon \log\left(\frac{1}{\varepsilon} + 1\right) \tag{5}$$

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for all $\varepsilon \in (0, T]$. Then there exists $K > 0$ such that

$$\begin{aligned} &\sup_{0\le t\le T} \{\|u(t,\cdot)\|_{\mathcal{H}^{s+1-K}} + \|\partial_t u(t,\cdot)\|_{\mathcal{H}^{s-K}}\} \\ &\qquad \le C_s(\|u(0,\cdot)\|_{\mathcal{H}^{s+1}} + \|\partial_t u(0,\cdot)\|_{\mathcal{H}^{s}} + \int_0^T \|Lu(t,\cdot)\|_{\mathcal{H}^{s}} \, dt), \end{aligned} \tag{6}$$

for all $u \in \mathcal{C}^2([0,T], \mathcal{H}^\infty(\mathbb{R}^n))$

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- The energy estimate (6) is deduced from the Fourier transform with respect to x of the equation together with an approximation of the coefficients which is different in different zones of the phase space (the so called *approximate energy technique)*
- The consequence of (6) is the $\mathcal{H}^\infty$–well–posedness *with loss of derivatives*. On the necessity of some kind of loss of derivatives when the regularity of the coefficients is measured with a modulus of continuity which is not the Lipschitz one, see [M. Cicognani and F. Colombini, J. Differential Eq. **221** (2006)].

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on the strip $[0, T] \times \mathbb{R}^n$.
L is *strictly hyperbolic* with bounded coefficients, i.e. there exist λ_0, $\Lambda_0 > 0$ such that

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It is well–known that if the coefficients a_{jk} are *Lipschitz–continuous* w.r.t. t (and only measurable w.r.t. x) then an *energy estimate* (in $\mathcal{H}^1$-$\mathcal{L}^2$) holds for L.

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Theorem (F. Colombini and N. Lerner, Duke Math. J. 77 (1995))

Suppose that there exists $C > 0$ *such that*

$$\sup_{\substack{y, y' \in [0,T] \times \mathbb{R}^n \\ |y'| = \varepsilon}} |(a_{jk}(y + y') - a_{jk}(y))| \leq C\varepsilon \log\left(\frac{1}{\varepsilon} + 1\right) \qquad (11)$$

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for all $\varepsilon \in (0,T]$. Then for all $\theta \in (0,1/4]$ there exist β, $C > 0$ and $T^* \in (0,T]$ such that

$$\begin{aligned} &\sup_{0\leq t\leq T^*} \{\|u(t,\cdot)\|_{\mathcal{H}^{-\theta+1-\beta t}} + \|\partial_t u(t,\cdot)\|_{\mathcal{H}^{-\theta-\beta t}}\} \\ &\leq C_\theta\Big(\|u(0,\cdot)\|_{\mathcal{H}^{-\theta+1}} + \|\partial_t u(0,\cdot)\|_{\mathcal{H}^{-\theta}} + \int_0^{T^*} \|Lu(t,\cdot)\|_{\mathcal{H}^{-\theta-\beta t}}\, dt\Big) \end{aligned} \tag{12}$$

for all $u \in \mathcal{C}^2([0,T^*],\mathcal{H}^\infty(\mathbb{R}^n))$.

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- The use of the Littlewood-Paley dyadic decomposition (in place of the Fourier transform with respect to *x*) together with the approximate energy technique are the crucial points in the proof.
- This result has been recently improved in [F. Colombini and G. Métivier, Ann. Sci. École Norm. Sup. (4) **41** (2008)] extending it to more general hyperbolic equations and systems, and deriving from it also a local uniqueness result and some applications to nonlinear hyperbolic equations.

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$$\sup_{(t,x)} |a_{jk}(t+\tau,x)+a_{jk}(t-\tau,x)-2a_{jk}(t,x)| \leq C_0\,|\tau| \log\left(\frac{1}{|\tau|}+1\right), \quad (13)$$

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Let $\theta \in (0,1)$. There exist T^*, β^*, $C > 0$ such that, for all $u \in \mathcal{C}^2([0,T], \mathcal{H}^\infty(\mathbb{R}^n))$, the following estimate holds:

$$\sup_{0 \le t \le T^*} \{ \|u(t,\cdot)\|_{\mathcal{H}^{-\theta+1-\beta^* t}} + \|\partial_t u(t,\cdot)\|_{\mathcal{H}^{-\theta-\beta^* t}} \}$$

$$\le C\Big(\|u(0,\cdot)\|_{\mathcal{H}^{-\theta+1}} + \|\partial_t u(0,\cdot)\|_{\mathcal{H}^{-\theta}} + \int_0^{T^*} \|Lu(t,\cdot)\|_{\mathcal{H}^{-\theta-\beta^* t}}\, dt \Big).$$

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Related results concerning operators depending on time and on *one* space variable are contained in [F. Colombini and D. D. S., J. Math. Sci. Tokyo **16**, No. 1, (2009)] and [F. Colombini and F. Fanelli, Rend. Istit. Mat. Univ. Trieste **42 Suppl.**, (2010)].

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$$a_\varepsilon(t,x) := \iint \rho_\varepsilon(t-s)\rho_\varepsilon(x-y)a(s,y)\,ds\,dy,$$

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Known facts on the Littlewood-Paley decomposition.
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We set $\varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi)$ and, if $\nu \ge 1$, $\varphi_\nu(\xi) = \varphi(2^{-\nu}\xi)$.
Let w be a tempered distribution; we define

$$w_\nu(x) := \varphi_\nu(D_x)w(x) = \frac{1}{2\pi}\int e^{ix\xi}\varphi_\nu(\xi)\hat{w}(\xi)\,d\xi = \frac{1}{2\pi}\int \hat{\varphi}_\nu(y)w(x-y)\,dy.$$

For all ν, w_ν is an entire analytic function belonging to $\mathcal{L}^2$ and for all $m \in \mathbb{R}$ there exists $K_m > 0$ such that

$$\frac{1}{K_m}\sum_{\nu=0}^{\infty}\|w_\nu\|^2_{\mathcal{L}^2}2^{2m\nu} \le \|w\|^2_{\mathcal{H}^m} \le K_m\sum_{\nu=0}^{\infty}\|w_\nu\|^2_{\mathcal{L}^2}2^{2m\nu}.$$

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Known facts on the Littlewood-Paley decomposition.
Let $\varphi_0 \in C_0^\infty(\mathbb{R}_\xi)$, $0 \leq \varphi_0(\xi) \leq 1$, $\varphi_0(\xi) = 1$ if $|\xi| \leq 1$, $\varphi_0(\xi) = 0$ if $|\xi| \geq 2$, φ_0 even and φ_0 decreasing on $[0, +\infty)$.
We set $\varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi)$ and, if $\nu \geq 1$, $\varphi_\nu(\xi) = \varphi(2^{-\nu}\xi)$.
Let w be a tempered distribution; we define

$$w_\nu(x) := \varphi_\nu(D_x)w(x) = \frac{1}{2\pi}\int e^{ix\xi}\varphi_\nu(\xi)\hat{w}(\xi)\,d\xi = \frac{1}{2\pi}\int \hat{\varphi_\nu}(y)w(x-y)\,dy.$$

For all ν, w_ν is an entire analytic function belonging to $\mathcal{L}^2$ and for all $m \in \mathbb{R}$ there exists $K_m > 0$ such that

$$\frac{1}{K_m}\sum_{\nu=0}^{\infty}\|w_\nu\|^2_{\mathcal{L}^2}2^{2m\nu} \leq \|w\|^2_{\mathcal{H}^m} \leq K_m \sum_{\nu=0}^{\infty}\|w_\nu\|^2_{\mathcal{L}^2}2^{2m\nu}.$$

Moreover, we have

$$2^{\nu-1}\|w_\nu\|_{\mathcal{L}^2} \leq \|\partial_x w_\nu\|_{\mathcal{L}^2} \leq 2^{\nu+1}\|w_\nu\|_{\mathcal{L}^2},$$

where the inequality on the right–hand side holds for all $\nu \geq 0$, while the other one holds only for all $\nu \geq 1$.

Proof in the case of one space variable: energy of the ν component

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Let $u(t, x)$ be a function in $C^2([0, T], \mathcal{H}^\infty(\mathbb{R}^n))$.
We set $u_\nu(t, x) = \varphi_\nu(D)u(t, x)$.
We obtain

$$\partial_t^2 u_\nu = \partial_x(a(t, x)\partial_x u_\nu) + \partial_x([\varphi_\nu, a]\partial_x u) + (Lu)_\nu.$$

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We introduce the approximate energy of u_ν, setting

$$e_{\nu,\varepsilon}(t) := \int_{\mathbb{R}} \frac{1}{\sqrt{a_\varepsilon}}\left|\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right|^2 + \sqrt{a_\varepsilon}|\partial_x u_\nu|^2 + |u_\nu|^2\,dx.$$

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Remark that in the case of Colombini and Lerner the approximate energy of the ν component was simply

$$e_{\nu,\varepsilon}(t) := \int_{\mathbb{R}} |\partial_t u_\nu|^2 + a_\varepsilon|\partial_x u_\nu|^2 + |u_\nu|^2\, dx.$$

Proof in the case of one space variable: energy of the ν component

We have

$$
\begin{aligned}
\frac{d}{dt} e_{\nu,\varepsilon}(t) = & \int \frac{2}{\sqrt{a_\varepsilon}} \mathrm{Re}\left(\partial_t^2 u_\nu \cdot \overline{(\partial_t u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu)} \right) dx \\
& + \int \frac{2}{\sqrt{a_\varepsilon}} \left(\partial_t (\frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}) - (\frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}})^2 \right) \mathrm{Re}\left(u_\nu \cdot \overline{(\partial_t u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu)} \right) dx \\
& + \int \partial_t \sqrt{a_\varepsilon} |\partial_x u_\nu|^2 \, dx + \int 2\sqrt{a_\varepsilon} \mathrm{Re}\,(\partial_x u_\nu \cdot \overline{\partial_x \partial_t u_\nu}) \, dx \\
& + \int 2 \mathrm{Re}\,(u_\nu \cdot \overline{\partial_t u_\nu}) \, dx.
\end{aligned}
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$$\frac{d}{dt}e_{\nu,\varepsilon}(t) = \int \frac{2}{\sqrt{a_\varepsilon}}\mathrm{Re}\left(\partial_t^2 u_\nu \cdot \overline{(\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu)}\right)dx$$

$$+\int \frac{2}{\sqrt{a_\varepsilon}}\left(\partial_t(\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}) - (\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}})^2\right)\mathrm{Re}\left(u_\nu\cdot\overline{(\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu)}\right)dx$$

$$+\int \partial_t\sqrt{a_\varepsilon}|\partial_x u_\nu|^2\,dx + \int 2\sqrt{a_\varepsilon}\mathrm{Re}\,(\partial_x u_\nu \cdot \overline{\partial_x\partial_t u_\nu})\,dx$$

$$+\int 2\mathrm{Re}\,(u_\nu \cdot \overline{\partial_t u_\nu})\,dx.$$

We replace $\partial_t^2 u_\nu$ by the quantity given by the equation and we obtain

Proof in the case of one space variable: energy of the ν component

$$\begin{aligned}\frac{d}{dt}e_{\nu,\varepsilon}(t) = &\int \frac{2}{\sqrt{a_\varepsilon}}\mathrm{Re}\left((\partial_x([\varphi_\nu, a]\partial_x u) + (Lu)_\nu)\cdot\overline{(\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu)}\right)dx\\
&+\int \frac{2}{\sqrt{a_\varepsilon}}\left(\partial_t(\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}) - (\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}})^2\right)\mathrm{Re}\left(u_\nu\cdot\overline{(\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu)}\right)dx\\
&+\int \partial_t\sqrt{a_\varepsilon}(\frac{a_\varepsilon - a}{a_\varepsilon})|\partial_x u_\nu|^2\,dx\\
&+\int 2(\frac{a_\varepsilon - a}{\sqrt{a_\varepsilon}})\,\mathrm{Re}\,(\partial_x u_\nu\cdot\overline{\partial_x\partial_t u_\nu})\,dx\\
&+\int 2\,\frac{\partial_x\sqrt{a_\varepsilon}}{a_\varepsilon}\,a\,\mathrm{Re}\left(\partial_x u_\nu\cdot\overline{(\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu)}\right)dx\\
&-\int \frac{a}{\sqrt{a_\varepsilon}}\,\partial_x(\frac{\partial_t\sqrt{a_\varepsilon}}{\sqrt{a_\varepsilon}})\,\mathrm{Re}\,(\partial_x u_\nu\cdot\overline{u_\nu})\,dx\\
&+\int 2\mathrm{Re}\,(u_\nu\cdot\overline{\partial_t u_\nu})\,dx.\end{aligned}$$

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$$\leq C_1\frac{1}{\varepsilon}\log(\frac{1}{\varepsilon}+1)\,2^{-\nu}\,e_{\nu,\varepsilon}(t)$$

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and

$$\Big| \int 2\big(\frac{a_\varepsilon - a}{\sqrt{a_\varepsilon}}\big) \mathrm{Re} \, (\partial_x u_\nu \cdot \overline{\partial_x \partial_t u_\nu}) \, dx \Big| \leq C_3'' \Big(\varepsilon \log\Big(\frac{1}{\varepsilon} + 1\Big) 2^\nu \, e_{\nu,\varepsilon}(t) \Big).$$

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We choose $\varepsilon = 2^{-\nu}$.

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$$\leq C_1 \frac{1}{\varepsilon} \log\big(\frac{1}{\varepsilon} + 1\big)\, 2^{-\nu}\, e_{\nu,\varepsilon}(t)$$

and

$$\Big| \int 2\big(\frac{a_\varepsilon - a}{\sqrt{a_\varepsilon}}\big) \mathrm{Re} \left(\partial_x u_\nu \cdot \overline{\partial_x \partial_t u_\nu} \right) dx \Big| \leq C_3'' \Big(\varepsilon \log \Big(\frac{1}{\varepsilon} + 1 \Big) 2^\nu\, e_{\nu,\varepsilon}(t) \Big).$$

We choose $\varepsilon = 2^{-\nu}$.
We obtain that there exists $\tilde{C} > 0$ such that, for all $\nu \in \mathbb{N}$,

$$\frac{d}{dt} e_{\nu,2^{-\nu}}(t) \leq \tilde{C}(\nu + 1) e_{\nu,2^{-\nu}}(t)$$

$$+ \int \frac{2}{\sqrt{a_{2^{-\nu}}}} \mathrm{Re} \Big(\big(\partial_x([\varphi_\nu, a]\partial_x u) + (Lu)_\nu \big) \cdot \overline{\big(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu \big)} \Big) dx.$$

Proof in the case of one space variable: total approximate energy

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Let $\theta \in (0, 1/4]$. We define the total energy for the function u setting

$$E(t) := \sum_{\nu=0}^{\infty} e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} e_{\nu,2^{-\nu}}(t),$$

where $\beta > 0$ will be fixed later on.

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where $\beta > 0$ will be fixed later on.
It is possible to prove that there exist c_θ, $c'_\theta > 0$ such that

$$E(0) \leq c_\theta (\|\partial_t u(0,\cdot)\|^2_{\mathcal{H}^{-\theta}} + \|u(0,\cdot)\|^2_{\mathcal{H}^{-\theta+1}})$$

and

$$E(t) \geq c'_\theta (\|\partial_t u(t,\cdot)\|^2_{\mathcal{H}^{-\theta-\beta^* t}} + \|u(t,\cdot)\|^2_{\mathcal{H}^{-\theta+1-\beta^* t}}),$$

where $\beta^* = \beta(\log 2)^{-1}$.

Proof in the case of one space variable: total approximate energy

We have

$$\frac{d}{dt}E(t) \leq (\tilde{C} - 2\beta)\sum_{\nu=0}^{\infty}(\nu+1)e^{-2\beta(\nu+1)t}2^{-2\nu\theta}e_{\nu,2^{-\nu}}(t) + \text{reminder}.$$

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$$\sum_{\nu=0}^{\infty} e^{-2\beta(\nu+1)t}2^{-2\nu\theta}\int \frac{2}{\sqrt{a_{2^{-\nu}}}}\mathrm{Re}\Big((Lu)_\nu \cdot \overline{\Big(\partial_t u_\nu + \frac{\partial_t\sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}}u_\nu\Big)}\Big)dx$$
$$\leq \tilde{C}_\theta E(t)^{1/2}\|Lu(t,\cdot)\|_{\mathcal{H}^{-\theta-\beta^* t}},$$

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$$\leq \tilde{C}_\theta E(t)^{1/2} \| Lu(t,\cdot) \|_{\mathcal{H}^{-\theta-\beta^* t}},$$

and a second term from the commutator $\partial_x([\varphi_\nu, a]\partial_x u)$ much more complicate to estimate.

Proof in the case of one space variable: the commutator term

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The term to estimate is

$$\sum_{\nu=0}^{\infty} e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} \int \frac{2}{\sqrt{a_{2^{-\nu}}}} \mathrm{Re}\Big(\partial_x([\varphi_\nu, a]\partial_x u) \cdot \overline{\Big(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\Big)}\Big) dx.$$

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Using Schur's lemma it is possible to prove that, choosing suitably βT^*, there exists a positive constant Γ_θ such that

$$\Big| \sum_{\nu=0}^{\infty} e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} \int \frac{2}{\sqrt{a_{2^{-\nu}}}} \mathrm{Re}\left(\partial_x([\varphi_\nu, a]\partial_x u) \cdot \overline{\left(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\right)}\right) dx \Big|$$
$$\leq \Gamma_\theta \sum_{\nu=0}^{\infty} (\nu+1) e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} e_{\nu, 2^{-\nu}}(t).$$

Proof in the case of one space variable: conclusion

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We have that

$$\frac{d}{dt}E(t) \leq (\tilde{C} + \Gamma_\theta - 2\beta)\sum_{\nu=0}^{\infty}(\nu+1)e^{-2\beta(\nu+1)t}2^{-2\nu\theta}e_{\nu,2^{-\nu}}(t)$$
$$+\tilde{C}_\theta E(t)^{1/2}\|Lu(t,\cdot)\|_{\mathcal{H}^{-\theta-\beta^* t}}.$$

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$$+\tilde{C}_\theta E(t)^{1/2}\|Lu(t,\cdot)\|_{\mathcal{H}^{-\theta-\beta^* t}}.$$

We fix now β in such a way that $\tilde{C} + \Gamma_\theta - 2\beta \leq 0$. Remark that since the product βT^* was fixed, this force us to choose T^* sufficiently small. We obtain

$$\frac{d}{dt}E(t) \leq \tilde{C}_\theta E(t)^{1/2}\|Lu(t,\cdot)\|_{\mathcal{H}^{-\theta-\beta^* t}},$$

and the conclusion follows.

Proof in the case of several space variables: paradifferential calculus

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In the case of Colombini and Lerner it is possible to pass from

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to

$$e_{\nu,\varepsilon}(t) := \int_{\mathbb{R}^n} |\partial_t u_\nu|^2 + \sum_{j,k}^{n} a_{j,k}^{\varepsilon} \partial_{x_j} u_\nu \overline{\partial_{x_k} u_\nu} + |u_\nu|^2 \, dx,$$

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Now we have adapt to several variables the energy

$$e_{\nu,\varepsilon}(t) := \int_{\mathbb{R}} |a_\varepsilon^{-\frac{1}{4}} \partial_t u_\nu + \partial_t (a_\varepsilon^{-\frac{1}{4}}) u_\nu|^2 + |a_\varepsilon^{\frac{1}{4}} \partial_x u_\nu|^2 + |u_\nu|^2 \, dx.$$

Proof in the case of several space variables: paradifferential calculus

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The idea is to use *paradifferential operators*.

Proof in the case of several space variables: paradifferential calculus

We define

$$a_\varepsilon(t,x,\xi) := \sum_{j,k}^{n} \Big(\int \rho_\varepsilon(t-s) a_{j,k}(s,x)\, ds \Big)\, \xi_j \xi_k$$

(remark that there is the regularization only w.r.t. t).

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(remark that there is the regularization only w.r.t. t).
We denote by T_{a_ε} the paradifferential operator associated to the symbol a_ε and by σ_{a_ε} its classical symbol and the same for $\partial_t a_\varepsilon$, $\partial_t^2 a_\varepsilon$ and the powers of a_ε.

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We denote by T_{a_ε} the paradifferential operator associated to the symbol a_ε and by σ_{a_ε} its classical symbol and the same for $\partial_t a_\varepsilon$, $\partial_t^2 a_\varepsilon$ and the powers of a_ε. It is possible to show that

$$|\partial_\xi^\alpha \sigma_{a_\varepsilon}(t,x,\xi)| \leq C_\alpha (1+|\xi|)^{2-|\alpha|},$$

$$|\partial_\xi^\alpha \partial_x^\beta \sigma_{a_\varepsilon}(t,x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{2-|\alpha|+|\beta|-1} \log(2+|\xi|).$$

and for σ_{a_ε} the usual paradifferential spectral conditions hold.

Proof in the case of several space variables: paradifferential calculus

Moreover

$$|\partial_\xi^\alpha \sigma_{\partial_t a_\varepsilon}(t,x,\xi)| \le C_\alpha \log^2(2+\frac{1}{\varepsilon})(1+|\xi|)^{2-|\alpha|},$$

$$|\partial_\xi^\alpha \partial_x^\beta \sigma_{\partial_t a_\varepsilon}(t,x,\xi)| \le C_{\alpha,\beta}\frac{1}{\varepsilon}(1+|\xi|)^{2-|\alpha|+|\beta|-1}\log(2+|\xi|),$$

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and

$$|\partial_\xi^\alpha \sigma_{\partial_t^2 a_\varepsilon}(t,x,\xi)| \le C_\alpha \frac{1}{\varepsilon}\log(2+\frac{1}{\varepsilon})(1+|\xi|)^{2-|\alpha|},$$

$$|\partial_\xi^\alpha \partial_x^\beta \sigma_{\partial_t^2 a_\varepsilon}(t,x,\xi)| \le C_{\alpha,\beta}\frac{1}{\varepsilon^2}(1+|\xi|)^{2-|\alpha|+|\beta|-1}\log(2+|\xi|),$$

and for both of them the usual spectral conditions hold.

Proof in the case of several space variables: paradifferential calculus

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and

$$|\partial_\xi^\alpha \sigma_{\partial_t^2 a_\varepsilon}(t,x,\xi)| \leq C_\alpha \frac{1}{\varepsilon}\log(2+\frac{1}{\varepsilon})(1+|\xi|)^{2-|\alpha|},$$

$$|\partial_\xi^\alpha \partial_x^\beta \sigma_{\partial_t^2 a_\varepsilon}(t,x,\xi)| \leq C_{\alpha,\beta}\frac{1}{\varepsilon^2}(1+|\xi|)^{2-|\alpha|+|\beta|-1}\log(2+|\xi|),$$

and for both of them the usual spectral conditions hold.
Similarly we define and check the behaviour of the paradifferential operators associated to the powers of a_ε.

Proof in the case of several space variables: energy of the ν-component

We set

$$e_{\nu,\varepsilon}(t) := \| T_{a_\varepsilon^{-1/4}} \partial_t u_\nu - T_{\partial_t(a_\varepsilon^{-1/4})} u_\nu \|^2_{\mathcal{L}^2} + \| T_{a_\varepsilon^{1/4}} u_\nu \|^2_{\mathcal{L}^2},$$

and the computations follow similarly to the previous case:

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- the paradifferential operator T_a is not positive if $a \geq \lambda_0 > 0$: we have to modify the definition of it.

Further results and open problems

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- Zygmund-continuity in t and x.
  Since Zygmund-continuity implies log-Lipschitz-continuity, in such a case the result of Colombini and Lerner ensures the well-posedness *with loss of derivatives*. Actually there is the well-posedness *without loss* in the case $\mathcal{H}^{\frac{1}{2}} \times \mathcal{H}^{-\frac{1}{2}}$ (work in progress). Is this result optimal?

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- Integral conditions vs pointwise conditions.
  In the case of coefficients depending only on time the regularity conditions are given as *integral conditions*. It is possible to use such a conditions also when the coefficients depend on x?

Thank you for your attention