“On the Montgomery-Hooley theorem in short intervals”

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Abstract: In 1970, dealing with with a problem studied by Barban, Davenport and Halberstam, Montgomery obtained the formula

$$S(x,Q) = xQ\log x + O\Big(xQ\log 2x/Q\Big) + O\Big(x^2\log^{-A} x\Big) \tag{1}$$

for $Q \le x$ and $x \to \infty$, where $A > 0$ is arbitrarily large and

$$S(x,Q) = \sum_{q\le Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} \left|\psi(x;q,a) - \frac{x}{\varphi(q)}\right|^2.$$

Such a result was subsequently improved by several mathematicians; the main contributions were achieved by Hooley, Croft, Friedlander-Goldston, Goldston-Vaughan and Vaughan.

In a joint paper with A. Perelli and A. Zaccagnini, we generalized Montgomery-Hooley asympotic formula (1) to the short interval case. To be more precise, letting

$$S(x,h,Q) = \sum_{q\le Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} \left|\psi(x+h;q,a) - \psi(x;q,a) - \frac{h}{\varphi(q)}\right|^2$$

and $\epsilon, A > 0$ be arbitrary, $x^{7/12+\epsilon} \le h \le x$ and $Q \le h$, we proved that

$$\begin{aligned} S(x,h,Q) &= hQ\log(xQ/h) + (x+h)Q\log(1+h/x) - \kappa hQ + O\Big(h^2\log^{-A} x\Big) \\ &\quad + O\Big(h^{1/2}Q^{3/2}\exp\Big(-c_1\frac{(\log 2h/Q)^{3/5}}{(\log\log 3h/Q)^{1/5}}\Big)\Big), \end{aligned}$$

where $\kappa = 1 + \gamma + \log 2\pi + \sum_p \frac{\log p}{p(p-1)}$ and γ is the Euler constant.

Assuming further that the Generalized Riemann hypothesis holds, and letting $\epsilon > 0$, $x^{1/2+\epsilon} \le h \le x$ and $Q \le h$, we proved that

$$\begin{aligned} S(x,h,Q) &= hQ\log(xQ/h) + (x+h)Q\log(1+h/x) - \kappa hQ + O\Big(\Big(\frac{h}{Q}\Big)^{1/4+\epsilon}Q^2\Big) \\ &\quad + O\Big(hx^{1/2}\log^{c_2} x\Big), \end{aligned}$$

for some positive constant c_2.

The quality of the error terms in our results are comparable with the ones obtained by Goldston-Vaughan in the classical case.