Surfaces with bounded integral curvature in the sense of Alexandrov

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Introduction

A story says that when Lebesgue was young, he once followed a class where Darboux was proving that any developable surface is ruled.

Lebesgue then threw a crumpled paper on the table:

Here is a developable surface. Where is the ruling ?

Darboux was doing differential geometry. Lebesgue observed that in nature, surfaces are rarely smooth.

Can we *unify* instead of *opposing* the smooth differential geometry of Darboux and the crumpled geometry of Lebesgue ? Is there a non trivial theory encompassing all surfaces one may reasonably conceive ?

In this talk, I would like to sketch the solution that A.D. Alexandrov and his students in Leningrad have proposed to this problem in the years 1940-1970.

Alexandre Danilovich Alexandrov (1912-1999)

Let us start with the following:

Problem 1.

(a) Is every smooth surface a limit of a sequence of polyhedral surfaces?

(b) Is every polyhedral surface a limit of a sequence of smooth surfaces?

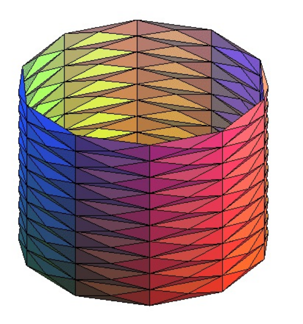
(c) What are the geometric invariant which converge under these limits?

A *smooth surface* is a 2-dimensional Riemannian manifold and a *polyhedral surface* is a metric space which is locally isometric to a 2-dimensional polyhedron.

The *Géode* of the *Cité des Sciences* in Paris is a beautiful concrete exmple of a polyhedral approximation of the round 2-sphere.

The area is not a continuous functional. In General:

$$\limsup_{i\to\infty} \mathrm{Area}(S_i) > \mathrm{Area}(\lim S_i).$$



The *H. A. Schwarz lantern* is a famous example of a sequence of (non convex) polyhedron P_i converging to an cylindre with

$$\lim_{i\to\infty} \mathrm{Aire}(P_i) = \infty$$

Let us reformulate our problem :

Problem 1'. Let us fix a topological surface S (say oriented and closed). We want to define a space $\mathcal{M}(S)$ containing all 'reasonable' metrics on S and we wish to endow $\mathcal{M}(S)$ with a topology for which both polyhedral metrics and smooth Riemannian metrics are dense subsets.

We also want to describe those geometric invariants which define continuous function on $\mathcal{M}(S)$.

Definition of Alexandrov surfaces

Definition A metric with bounded integral curvature on S in the sense of Alexandrov is a continuous function

$$d : S \times S \to \mathbb{R}$$

such that

(i) d is a distance and this distance induces the manifold topology on S;

(ii) the metric d is *geodesic*, i.e. every pair of points $x, y \in S$ can be joined by a curve of length $\ell = d(x, y)$;

(iii) the distance d on S is a uniform limit of a sequence of distances associated to Riemannian metrics on S for which the absolute value of the curvature is uniformly bounded.

We shall explain the third condition.

Definition (uniform distance)

Denote by $\mathrm{Met}(S)$ the set of metrics on S satisfying conditions (i) and (ii). The *uniform distance* between $d_1, d_2 \in \mathrm{Met}(S)$ is defined by

$$D(d_1, d_2) = \sup\{|d_1(x,y) - d_2(x,y)| \,:\, x, y \in S\}.$$

A review on Riemannian surfaces

The metric $d \in \mathrm{Met}(S)$ is *Riemannian* if a smooth structure is given on S as well as a positive definite 2-tensor $g \in \Gamma(S^2 T^*(S))$ such that

$$d(x, y) = \inf \int_0^1 \sqrt{g(\dot\gamma(t), \dot\gamma(t))}dt,$$

where the infimum is taken on all path $\gamma : [0, 1] \to S$ joining x to y.

On any small domain $U \subset S$, on can find a moving coframe $\theta^1, \theta^2 \in \Omega^1(U)$ such that $\theta^1 \wedge \theta^2 > 0$ and

$$g = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2.$$

the *Hodge star operator* $* : \Omega^k(S) \to \Omega^{2-k}(S)$ $(k = 0, 1, 2)$ is defined by the conditions

$$*(1) = \theta^1 \wedge \theta^2, \qquad *(\theta^1 \wedge \theta^2) = 1, \qquad *\theta^1 = \theta^2, \qquad *\theta^2 = -\theta^1$$

The *connexion form* associated to the coframe θ^1, θ^2 is the 1-form $\omega \in \Omega^1(U)$ given by

$$\omega = -(*d\theta^1)\theta^1 - (*d\theta^2)\theta^2, \tag{1}$$

it is the unique 1-form on U such that the Elie Cartan structure equations

$$\begin{cases} d\theta^1 & = -\omega \wedge \theta^2 \\ d\theta^2 & = \ \ \omega \wedge \theta^1 \end{cases}$$

hold.

Ww then define the *area measure* ant the *curvature measure* by

$$dA = \theta^1 \wedge \theta^2 \qquad \text{and} \qquad d\omega$$

We easily check that dA and $d\omega$ do not depend on the chosen coframe $\theta^1, \theta^2 \in \Omega^1(U)$ and are thus globally defined 2-forms (caution: these forms are not exact !).

Gauss-Bonnet Formula

$$\int_S d\omega = 2\pi\chi(S).$$

The *Gauss curvature* $K : S \to \mathbb{R}$ is the Radon Nikodym derivative of $d\omega$ against dA:

$$d\omega = KdA \qquad \Leftrightarrow \qquad K = *d\omega$$

We also define

$$d\omega^+ = K^+ dA_g, \qquad d\omega^- = K^- dA_g \quad \text{et} \quad d|\omega| = |K| dA_g,$$

(where $K^+ = \max\{K, 0\}$ and $K^- = \max\{-K, 0\}$).

Observe that

$$d\omega = d\omega^+ - d\omega^-, \qquad d|\omega| = d\omega^+ + d\omega^-.$$

We can now complete our definition :

Definition A *metric with bounded curvature in the sense of Alexandrov* on S is a continuous function

$$d : S \times S \to \mathbb{R}$$

such that

(i) d is a distance and this distance defines the manifold topology on S;

(ii) the metric d is *geodesic*, i.e. every pair of points $x, y \in S$ can be joined by a curve of length $\ell = d(x, y)$;

(iii) the distance d on S is a uniform limit of a sequence of distances associated to Riemannian metrics on S for which the absolute value of the curvature is uniformly bounded.

Condition (iii) says that there exists a sequence of smooth Riemannian metrics $\{g_j\}$ on S such that

$$D(d_{g_j}, d) \to 0$$

and

$$\sup_{j\in\mathbb{N}} \int_S d\omega_j^+ < \infty.$$

The Gauss-Bonnet formula implies that $\int_S d|\omega_j|$ is also bounded. By compactness, there exists then a Radon measure $d\omega$ on S and a subsequence $\{g_{j'}\}$ of $\{g_j\}$ such that

$$d\omega_{g_{j'}} \to d\omega \qquad \text{(weakly)}$$

Theorem

The Radon measure $d\omega = \lim_{j'} d\omega_{g_{j'}}$ is well defined on the metric space (S, d), it is independent of the chosen sequence $\{g_j\}$.

Definition The limit measure $d\omega$ is called the *curvature measure* of the Alexandrov surface (S, d).

Alexandrov shows that we can construct the measure $d\omega$ directly from the metric d on$S$ without referring to the Riemannian approximation.

The idea of the construction is to triangulate the surface by (small) geodesic triangles and measure in each triangle the angular excess.

A glueing construction

Let us consider a finite collection $\{T_1, T_2, \dots, T_m\}$ Riemannian triangles, i.e. manifolds homeomorphic to a disk equipped with a C^2 Riemannian metric with 3 corners.

Let us glue these triangles pairwise according to a prescribed triangulation of our surface S. Two adjacent sides being identified by an isometry.
We thus obtain a length space homeomorphic to S which is an Alexandrov surface whose curvature measure is given by

$$d\omega = d\omega_0 + d\omega_1 + d\omega_2,$$

where $d\omega_2$ is absolutely continuous with respect to the area measure and is given by

$$d\omega_2 = KdA$$

in the interior of each triangle.

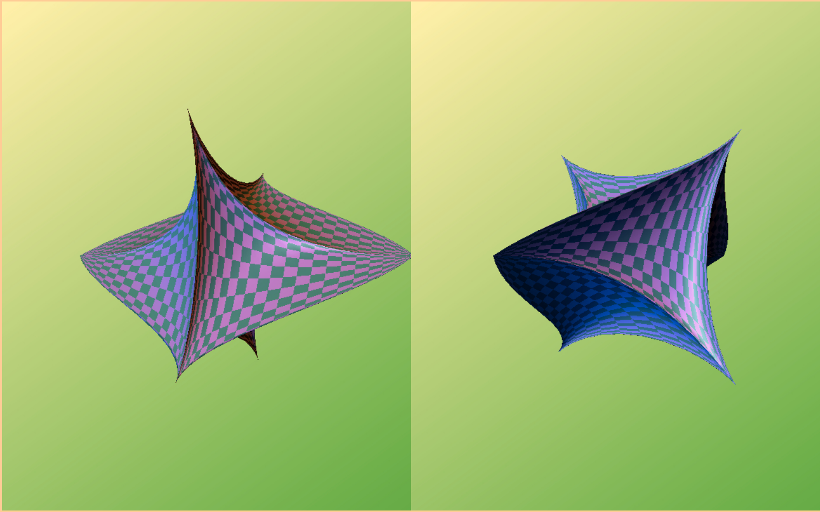
The measure $d\omega_0$ is a discrete measure supported by the vertices of the triabgulation and such that for each vertex p, one has

$$\omega_0(\{p\}) = 2\pi - (\text{sum of the angles of all } T_i \text{ incident with } p).$$

And $d\omega_1$ is supported by the edges of the triangulation, on each edge $a = T_i \cap T_j$, we have

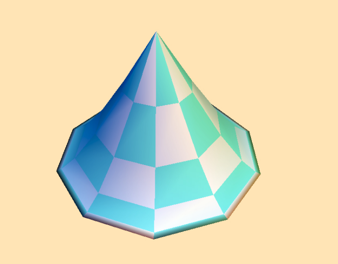
$$d\omega_1 = (k^+ - k^-)ds$$

where k^+ et k^- are the geodesic curvatures of a seen in each adjacent triangle and coherently oriented.



Polyhedral surfaces

As a special case: a *polyhedral surface* is a surface which can be obtained by gluing euclidean flat triangles.



The curvature measure is then concentrated on the vertices and near any vertex, the surface is locally isometric with an euclidean cone.

For this reasons, polyhedral surfaces are also called *flat surfaces with conical singularities*. They are exactly those Alexandrov surfaces whose curvature measure is discrete.

Exemple : the cube

Let us consider the surface of a cube. The faces are flat and thus $d\omega_2 = 0$, the edges are geodesic within both of their adjacent sides, hence $d\omega_1 = 0$. The eight vertices are each incident with 3 angles of $\frac{\pi}{2}$ radians, we thus have

$$\omega(p) = 2\pi - 3\frac{\pi}{2} = \frac{\pi}{2}$$

at each vertex.

We easily check Gauss-Bonnet:

$$\int_S d\omega = \int_S d\omega_0 = 8 \times \frac{\pi}{2} = 4\pi = 2\pi\chi(S^2).$$

Exemple 2 : a cylindre

(courtesy of Andy Warhol)

A food can is an euclidean cylinder of radius r and height h together with its top and its bottom which are euclidean disks D_1, D_2 of radius r. The cylinder and both disks are flat, thus the curvature is concentrated on the circles bounding the disks. These circles are geodesics in the cylinder and they have constant geodesic curvature $k = \frac{1}{r}$ in the disks D_i.

We thus have $d\omega_0 = d\omega_2 = 0$ and

$$d\omega_1 = \frac{1}{r}\, ds|_{\partial D_1} + \frac{1}{r}\, ds|_{\partial D_2}\,.$$

We again check Gauss-Bonnet:

$$\int_S d\omega = \int_S d\omega_1 = \frac{1}{r}\text{Length}(\partial D_1) + \frac{1}{r}\text{Length}(\partial D_2) = \frac{1}{r}(2\pi r + 2\pi r) = 4\pi.$$

Conformal structures of smooth surfaces

Let us return to smooth surfaces

Definitions a Riemannian metric $\tilde{g}$ on S is a *conformal deformation* of g if there exists a function $u : S \to \mathbb{R}$ such that

$$\tilde{g} = e^{2u} g.$$

If u is constant, on says that $\tilde{g}$ is *homothetic* to g.

A local coordinate system (x, y) on an open set $U \subset S$ is *conformal* (isothermal) for the metric g if there exists a function $\rho : U \to \mathbb{R}$ such that

$$g = \rho(x, y)(dx^2 + dy^2).$$

Definition. The *Laplacien* of u with respect to the metric g is

$$\Delta_g u = - * d * du,$$

i.e. $\Delta_g u \, dA = -d * du$.

It is in elliptic operator, it writes in coordinate as

$$\Delta_g u = -\frac{1}{\sqrt{\det(g_{ij})}} \sum_{\mu,\nu=1}^{2} \frac{\partial}{\partial x_\mu} \left(g^{\mu\nu} \sqrt{\det(g_{ij})} \cdot \frac{\partial u}{\partial x_\nu} \right).$$

In conformal coordinates, if $g = \rho(x,y)(dx^2 + dy^2)$ then

$$\Delta_g u = -\frac{1}{\rho(x,y)} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

Proposition

Every smooth Riemannian metric on a surface S admits conformal coordinates in a neighborhood of any point.

Proof. Let $U \subset S$ be an open set where a moving coframe $\theta^1, \theta^2 \in \Omega^1(U)$ is defined, i.e. $g = (\theta^1)^2 + (\theta^1)^2$. Then $\tilde{\theta}^1 = e^u \theta^1$, $\tilde{\theta}^2 = e^u \theta^2$ is a coframe for the metric $\tilde{g} = e^{2u} g$, and the connection forms ω et $\widetilde{\omega}$ are related by

$$\widetilde{\omega} = \omega - *du.$$

Thus

$$d\widetilde{\omega} = d\omega - d * du. \tag{2}$$

But we clearly have

$$dA = \theta^1 \wedge \theta^2 \qquad d\widetilde{A} = \widetilde{\theta}^1 \wedge \widetilde{\theta}^2 = e^{2u} dA$$

The equation (2) thus writes

$$\tilde{K} e^{2u} = K + \Delta_g u. \tag{3}$$

Choose a local solution of $u : U \to \mathbb{R}$ such that $\Delta_g u = -K$, then $\tilde{g} = e^{2u} g$ is flat and we can thus find euclidean coordinates x, y around each point of U. In such coordinates $\tilde{g} = dx^2 + dy^2$ and therefore

$$g = e^{-2u}(dx^2 + dy^2).$$

$\square$

Corollary

Every Riemannian metric g on a smooth oriented surface defines a complex structure on that surface ($=$ structure of Riemann surface).

Theorem (Poincare-Koebe Uniformization theorem)

Every Riemannian metric g on a smooth oriented surface is a conformal deformation of a metric h with constant curvature

Proof. Solve

$$\tilde{K}e^{2u} = K + \Delta_g u.$$

With $K = +1, -1, 0$ according to the sign of $\chi(S)$.

Green function and potential

The inverse of the laplacien is given by the Green function:

Theorem

Let (S,h) be a closed smooth Riemannian surface. There exists a unique function $G : S \times S \to \mathbb{R} \cup \{+\infty\}$ such that

(a) *G is C^∞ on $S \times S \setminus \{(x,x) \,|\, x \in S\}$;*

(b) *$G(x,y) = G(y,x)$;*

(c) *$|G(x,y)| \leq C \cdot (1 + |\log d(x,y)|)$;*

(d) *$\int_S G(x,y)dA_h(y) = 0$;*

(e) *For any $u \in C^2(S)$, we have*

$$u(x) = \int_S G(x,y)\Delta u(y)dA_h(y) + \frac{1}{area(S)} \int_S u(y)dA_h(y).$$

Proposition

If μ is a Radon (signed) measure on S such that $\int_S d\mu = 0$, then the function

$$u(x) = \int_S G(x,y)d\mu(y)$$

satisfies

$$\Delta u = \mu$$

in the weak sense.

Definition The function u is called the *potential* of the Radon measure $d\mu$ with respect to the metric h.

We note $V(S,h)$ the space of functions $u \in L^1(S)$ such that $\mu = \Delta_h u$ is a measure. For $u \in V(S,h)$ and $x, y \in S$, we set

$$d_{h,u}(x,y) = \inf \left\{ \int_0^1 e^{u(\alpha(t))} |\dot{\alpha}(t)|_h dt \,\Big|\, \alpha \in \mathcal{C}(x,y) \right\} \qquad (4)$$

where $\mathcal{C}(x,y)$ is the set of C^1-paths $\alpha : [0,1] \to S$ joining x to y.

Then $d_{h,u}$ is a pseudo-metric, i.e. $d_{h,u}(x,y)$ is symetric and satisfies the triangular inequality.

Furthermore $0 \leq d_{h,u}(x,y) \leq \infty$ for any $x, y \in S$.

Proposition (Reshetnyak)

The pseudo-metric $d_{h,u}$ is separating, i.e. $d_{h,u}(x, y) > 0$ if $x \neq y$. Furthermore $d_{h,u}(x, y) < \infty$ for any pair of points $x, y \in S$ such that $\mu(\{x\}) < 2\pi$ and $\mu(\{y\}) < 2\pi$ where μ is the measure $\Delta_h u$.

We shall see later that $(S, d_{h,u})$ is a surface with bounded integral curvature in the sense of Alexandrov

Summary: If (S, h) is a closed Riemannian surface and μ is a Radon measure on S of zero integral and such that $\mu(\{x\}) < 2\pi$ for any $x \in S$, and u is the potential of μ, then $g = e^{2u}h$ is a singular Riemannian metric on S for which the associated pseudo-distance is a true distance.

Remark A point x such that $\mu(\{x\})) = \Delta_h u(\{x\}) = 2\pi$ gives rise to *cusp*. It can (bust must not) be an infinite distance away.

Convergence theorems

The curvature measure of an Alexandrov depends continuously on the metric:

Theorem (Alexandrov)

Let (S, d) be a closed surface with bounded integral curvature in the sense of Alexandrov. If $\{d_j\}$ is a sequence of metrics with bounded integral curvature converging to d in the uniform topology, then the curvature measure of (S, d) is the weak limit of the curvature measures of $\{(S, d_j)\}$, i.e.

$$D(d_j, d) \to 0 \implies d\omega_j \xrightarrow{\text{weakly}} d\omega$$

The converse also holds, provides the conformal structure be fixed and no cusp be allowed:

Theorem (Reshetnyak, 1960)

Let (S, h) be a closed smooth Riemannian surface $\{d\mu_n^+\}$, $\{d\mu_n^-\}$ two sequences of Radon measures on S converging weakly to $d\mu^+ = \lim_{n\to\infty} d\mu_n^+$ and $d\mu^- = \lim_{n\to\infty} d\mu_n^-$. Assume that $\mu^+(\{p\}) < 2\pi$ for any point $p \in S$. Let u_n be the potential of $d\mu_n = d\mu_n^+ + d\mu_n^-$ and u be the potential of $d\mu = d\mu^+ + d\mu^-$, then

$$d_{h,u_n} \longrightarrow d_{h,u}$$

in the uniform topology.

Corollary

Let (S,h) be a closed smooth Riemannian surface and μ be a Radon measure on S such that $\int_S d\mu = 0$ and $\mu(\{x\}) < 2\pi$ for any $x \in S$. Let u be the potential of μ.
Then the metric $d_{h,u}$ has bounded integral curvature in the sense of Alexandrov.
The curvature measure of $(S, d_{h,u})$ is

$$d\omega = K_h dA_h + d\mu. \tag{5}$$

Proof Choose a sequence of smooth measures $d\mu_j = \varphi_j dA_h$ converging to $d\mu$ and such that $\int_S d\mu_j = 0$. Let u_j be the potential of $d\mu_j$, then $g_j = e^{2u_j}h$ is a smooth Riemannian metric and the associated distances d_j converge to $d_{h,u}$ for the uniform topology by the Reshetnyak theorem. Thus $(S, d_{h,u})$ is a surface with bounded integral curvature in the sense of Alexandrov by definition. The identity (5) follows then from (3) by continuity.

The conformal structure is determined by the metric:

Theorem (Conformal rigidity)

Let (S, h) et (S', h') be two smooth Riemannian surfaces and $u \in V(S, h)$, $u' \in V(S', h')$.
Denote by $d_{h,u}$ and $d_{h',u'}$ the Alexandrov metrics correponding to $e^u h$ and $e^{u'} h'$.
Then any isometry $f : (S, d_{h,u}) \to (S', d_{h',u'})$ is a conformal transformation from (S, h) to (S', h').

The proof follows from a theorem of Menchoff (1937) which says that any 1-quasi-conformal map is conformal.

Conformal structure of Alexandrov surfaces

Reshetnyak (and Huber) has deduced from his theorem that any metric with bounded integral curvature on a surface S determines a unique conformal structure on S

Theorem

Let (S, d) be an Alexandrov surface without cusp. Then there exists a smooth Riemannian metric h and a function $u \in V(S, h)$ such that

$$d = d_{h,u}.$$

Corollary

Any closed oriented surface (S, d) with bounded integral curvature without cusp determines a unique complex structure on S.

Proof. By the previous theorem, one can write d as $d = d_{h,u}$ where h is Riemannian and $u \in V(S, h)$. The result on conformal (isothermal) coordinates allows us to introduce a well defined complex structure on S and the conformal rigidity theorem says that this complex structure depends on the metric d only.

$\square$

Summary: A metric with bounded integral curvature without cusp on a closed oriented surface S determines the following data:

(i) A conformal structure on S;

(ii) The curvature measure $d\omega$.

Conversely:

Theorem

For any conformal structure on S and any Radon measure $d\omega$such that $\int_S d\omega = 2\pi\chi(S)$ and $\omega(\{x\}) < 2\pi$ for any x, there exists an Alexandrov metric on S associated to this conformal structure and whose curvature measure is given by $d\omega$.

This Alexandrov metric is unique up to homothety.

Proof. Fix a conformal structure on S and represent it by a smooth Riemannian metric h with constant curvature. Let $d\mu = d\omega - K_h dA_h$ and $u \in V(S, h)$ be the potential of $d\mu$. Then the metric $d_{h,u}$ has the desired properties.

To prove uniqueness, consider another Alexandrov metric d' on S. By Reshetnyak's theorem, there exists a Riemannian metric h' on S and $u' \in V(S', h')$ with $d' = d_{h',u'}$. The conformal Rigidity implies that h and h' are conformally equivalent, i.e. there exists $v \in C^\infty(S)$ such that $h' = e^{2v}h$. Replacing u' by $u' + v$ if necessary, one may assume that $h = h'$.
We thus have $d' = d_{h,u'}$ with curvature measure $d\omega$, thus

$$\Delta_h u' = d\omega - K_h dA_h = \Delta_h u.$$

Therefore $\Delta_h(u' - u) = 0$ and $(u' - u)$ must be constant.

$\square$

This results can be seen as a classification theorem for Alexandrov surfaces .

Let S be an oriented closed surface. Denote by $\mathcal{M}_0(S)$ the space of Alexandrov metrics on S without cusps, $\mathcal{C}(S)$ the space of conformal structures and $\mathcal{R}_{2\pi}(S)$ the space of Radon measures $d\omega$ on S such that

$$\int_S d\omega = 2\pi\chi(S) \qquad \text{and} \qquad \omega(\{x\}) < 2\pi \ \text{ for all } x \in S.$$

The preceeding theorem says that

$$\mathcal{M}_0(S) = \mathcal{C}(S) \times \mathcal{R}_{2\pi}(S) \times \mathbb{R}_+ .$$

(where $\mathbb{R}_+$ controls the homothety factor).

Note that the space of Radon measure on S is metrizable and locally compact, and that the space of conformal structures is well understood via the viewpoint of Teichmüller theory.

Corollary

Let S be a closed surface, $x_1, x_2, \cdots, x_n$ be points on S and $\theta_1, \theta_2, \cdots \theta_n > 0$. Assume $\sum_i (2\pi - \theta_i) = 2\pi\chi(S)$, then for any conformal structure on S, there exists a polyhedral metric on S with conical singularity of cone angle θ_i at x_i $(i = 1, \cdots, n)$. This metric is unique up to homothety

Proof. Apply the previous theorem to the discrete measure

$$d\omega = \sum_i (2\pi - \theta_i)\delta_{x_i}.$$

$\square$

More generally:

Corollary

Any Radon measure $d\omega$ on S such that $\int_S d\omega = 2\pi\chi(S)$ and $\omega(\{x\}) < 2\pi$ for all x is the curvature measure of an Alexandrov metric.

Corollary

Every Alexandrov surface is a limit of a sequence of polyhedral surfaces.

Proof. Approximate the curvature measure of the given Alexandrov surface by a sequence of discrete measures.

□

Recall that, by definition, every Alexandrov surface is a limit of a sequence of smooth Riammanian surfaces.

Grazie per la Vostra attenzione !